$\mathcal{C}$osmo$\mathcal{L}$attice

A modern code for lattice simulations of scalar
and gauge field dynamics in an expanding universe

– Technical Note III –
Gravitational Waves from U(1) Gauge Theories

Written on June 16, 2023

Authors

Jorge Baeza-Ballesteros[*]
Daniel G. Figueroa[†]
Nicolás Loayza[‡]

Code Developers

Jorge Baeza-Ballesteros
Nicolás Loayza

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[*]jorge.baeza@ific.uv.es

[†]daniel.figueroa@ific.uv.es

[‡]nicolas.loayza@ific.uv.es

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Gravitational Waves from U(1) Gauge Theories
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Jorge Baeza-Ballesteros, Daniel G. Figueroa and Nicolás Loayza

Instituto de Física Corpuscular (IFIC), Consejo Superior de Investigaciones Científicas (CSIC) and Universitat de Valencia (UV), 46980,Valencia, Spain.

Abstract

This is a *technical note* about the dynamics of gravitational waves (GWs) sourced by U(1) charged complex scalars and Abelian gauge fields in a lattice. All new features are implemented in the (updated) *GW module* that forms part of $\mathcal{C}osmo\mathcal{L}attice$ v1.2, which is publicly available in http://www.cosmolattice.net. We recommend the reader to check out as well other *technical notes* available there.

Important Note: If the reader is not familiar with the concept of gravitational waves in a lattice, we recommend them to read first our Technical Note II [1], for a more detailed introduction of the relevant concepts.

1 Gravitational waves in the continuum

We assume the reader is familiar with the concept of gravitational waves (GWs) in the continuum. Nevertheless, we review briefly the definition of GWs in a spatially-flat Friedman-Lemaître-Robertson-Walker (FLRW) metric. This allows us to remind the reader of the strategy for numerically solving the GW dynamics and to introduce the field variables we need in an Abelian gauge theory. GWs are identified with perturbations h_{ij} of the background metric which are transverse and traceless (TT), i.e.,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j\,, \quad \text{with} \ \ \partial_i h_{ij} = 0 \text{ and } h_{ii} = 0\,, \tag{1}$$

where a is the scale factor, t represents coordinate time and x^i are spatial coordinates, with latin ndices running from 1 to 3. Throughout this note, summation is assumed over repeated indices, unless otherwise stated. In a FLRW background, the dynamics of GWs are described by equations of motion of the form [2]

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = \frac{2}{m_{\mathrm{p}}^2 a^2}\Pi_{ij}^{\mathrm{TT}}\,, \tag{2}$$

where $\dot{h}_{ij} = \mathrm{d}h_{ij}/\mathrm{d}t$, $H = \dot{a}/a$ is the Hubble rate, $m_{\mathrm{p}} = 1/\sqrt{8\pi G} = 2.44 \times 10^{18}$ GeV is the reduced Planck mass and Π_{ij}^{TT} is the transverse-traceless (TT) part of the anisotropic tensor Π_{ij}. The conditions $\partial_i\Pi_{ij}^{\mathrm{TT}} = \Pi_{ii}^{\mathrm{TT}} = 0$ hold $\forall\,\mathbf{x}, t$. Obtaining the TT part of a tensor is a non-local operation, and so is computationally very expensive. Thus, instead of solving directly Eq. (2), we follow the procedure introduced in Ref. [7]. In particular, we write the TT tensor perturbations as

$$h_{ij}(\mathbf{k}, t) = \Lambda_{ij,kl}(\hat{\mathbf{k}})u_{kl}(\mathbf{k}, t)\,, \tag{3}$$

where $\Lambda_{ij,kl}$ is a *projection operator* defined in Fourier space as

$$\Lambda_{ij,lm}(\hat{\mathbf{k}}) \equiv P_{il}(\hat{\mathbf{k}})P_{jm}(\hat{\mathbf{k}}) - \frac{1}{2}P_{ij}(\hat{\mathbf{k}})P_{lm}(\hat{\mathbf{k}})\,, \quad \text{with} \quad P_{ij}(\boldsymbol{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j\,, \quad \hat{k}_i = k_i/k\,, \tag{4}$$

with $\boldsymbol{k}$ is the momentum. In Eq. (2) the functions $u_{ij}(\mathbf{k},t)$ are the Fourier transform of the solution to

$$\ddot{u}_{ij} + 3H\dot{u}_{ij} - \frac{\nabla^2}{a^2}u_{ij} = \frac{2}{m_{\rm p}^2 a^2}\Pi_{ij}^{\rm eff}\,, \tag{5}$$

where $\Pi_{ij}^{\rm eff}$ is an *effective* anisotropic stress tensor that contains only those contributions from Π_{ij} which have non-vanishing TT-projection.

Consider now a theory with one real singlet field and one complex scalar field charged under an Abelian U(1) gauge symmetry, and the corresponding gauge boson. The action is

$$S = -\int d^4x\sqrt{-g}\mathcal{L} = -\int dx^4\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\phi,|\varphi|)\right\}\,, \tag{6}$$

where $\mathcal{L}$ is the Lagrangian of the model, $g = \det(g_{\mu\nu})$ is the determinant of the metric and

$$\phi \in \mathbb{R}\,, \quad \varphi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \in \mathbb{C}\,, \quad \text{with} \quad \phi_1,\phi_2 \in \mathbb{R}\,. \tag{7}$$

The U(1) gauge field stress tensor and gauge covariant derivative are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad \text{and} \quad D_\mu\varphi = \partial_\mu\varphi - ig_A Q_A A_\mu\varphi\,, \tag{8}$$

where g_A is the gauge coupling and Q_A is the U(1) charge of φ. From the Lagragian in Eq. (6), we can write down the stress-energy tensor of the scalar and gauge fields.

$$\begin{aligned}
T_{\mu\nu} &\equiv -\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L})}{\delta g^{\mu\nu}} = g_{\mu\nu}\mathcal{L} - 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \\
&= -g_{\mu\nu}\left(g^{\alpha\beta}\left[(D_\alpha\varphi)^*(D_\beta\varphi) + \frac{1}{2}(\partial_\alpha\phi)(\partial_\beta\phi)\right] + \frac{1}{4}g^{\alpha\delta}g^{\beta\lambda}(F_{\alpha\beta}F_{\delta\lambda}) + V(\phi,\varphi)\right) \\
&\quad + (\partial_\mu\phi)(\partial_\nu\phi) + [2\mathrm{Re}\{(D_\mu\varphi)^*(D_\nu\varphi)\} + g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}]\,.
\end{aligned} \tag{9}$$

From here, one can determine with a bit of work [3] the effective source of Eq. (5), which contains only those parts of Π_{ij} that do not vanish after TT-projecting, namely

$$\Pi_{ij}^{\rm eff} = \partial_i\phi\,\partial_j\phi + 2\mathrm{Re}\{(D_i\varphi)^*(D_j\varphi)\} - \left(a^{-2\alpha}E_iE_j + a^{-2}B_iB_j\right)\,, \tag{10}$$

where E_i and B_i are the Abelian electric and magnetic fields, defined as

$$E_i \equiv F_{0i}\,, \quad \text{and} \quad B_i \equiv \frac{1}{2}\epsilon_{ijk}F^{jk}\,. \tag{11}$$

2 Gravitational waves in $\mathcal{C}$osmo$\mathcal{L}$attice

In order to numerically study the dynamics of the fields, one needs to work with dimensionless quantities, also known as *program variables*. In $\mathcal{C}$osmo$\mathcal{L}$attice they are defined as

$$\tilde{\phi}_a = \frac{\phi_a}{f_*}\,, \quad d\tilde{\eta} = a^{-\alpha}\omega_* dt\,, \quad d\tilde{x}^i = \omega_* dx^i\,, \quad \kappa_i = \frac{k_i}{\omega_*}\,, \quad \tilde{A}_\mu = \frac{A_\mu}{\omega_*}\,, \tag{12}$$

where ϕ_a refers to any scalar field (real or complex), A_μ to a Abelian gauge field, and α, f_* and ω_* are constants. The last two have dimensions of energy, whereas α is dimensionless. Their particular values should be chosen based on the model that one wishes to simulate, see Ref. [5] for a detailed discussion. We denote the time derivative with respect to program time by $' = d/d\tilde{\eta}$ and the gradient $\tilde{\nabla}_i = d/d\tilde{x}^i$. The source of gravitational waves is the transverse-traceless part $\Pi_{ij}^{\rm TT}$ of the anisotropic stress-energy tensor. For the case of the u fields we can write down their source as the effective anisotropic tensor tensor,

$$\tilde{\Pi}_{ij}^{\rm eff} = \frac{\Pi_{ij}^{\rm eff}}{\omega_*^2 f_*^2} = \tilde{\nabla}_i\tilde{\phi}\,\tilde{\nabla}_j\tilde{\phi} + 2\mathrm{Re}\{(\tilde{D}_i\tilde{\varphi})^*(\tilde{D}_j\tilde{\varphi})\} - \left(\frac{\omega_*^2}{f_*^2}\right)[a^{-2\alpha}\tilde{E}_i\,\tilde{E}_j + a^{-2}\tilde{B}_i\,\tilde{B}_j]\,, \tag{13}$$

where $\tilde{E}_i$ and $\tilde{B}_i$ are the Abelian electric and magnetic fields, in program units, defined as

$$\tilde{E}_i \equiv \frac{E_i}{\omega_*^2} \quad \text{and} \quad \tilde{B}_i \equiv \frac{B_i}{\omega_*^2} \; . \tag{14}$$

and $\tilde{D}_i = \tilde{\nabla}_i - ig_A Q_a \tilde{A}_i$. In $\mathcal{C}$osmo$\mathcal{L}$attice, we define normalized $\tilde{u}$ fields as

$$h_{ij}(\mathbf{k}, t) = \Lambda^{\mathrm{L}}_{ij,kl}(\hat{\mathbf{k}}) u_{kl}(\mathbf{k}, t) = \left(\frac{f_*}{m_{\mathrm{p}}}\right)^2 \Lambda^{\mathrm{L}}_{ij,kl}(\hat{\mathbf{k}}) \tilde{u}_{kl}(\mathbf{k}, t) \; , \tag{15}$$

and their conjugate momentum as $(\pi_{\tilde{u}})_{ij} = a^{3-\alpha} \tilde{u}'_{ij}$. The lattice definition of $\Lambda^{\mathrm{L}}_{ij,kl}$ depends on the particular choice of the lattice momenta, see Ref. [1] for a detailed discussion. This allows to rewrite Eq. (5) as a system of first order differential equations

$$\begin{cases} \tilde{u}'_{ij} = a^{\alpha-3}(\pi_{\tilde{u}})_{ij} \; , \\ (\pi_{\tilde{u}})'_{ij} = a^{1+\alpha}\tilde{\nabla}^2 \tilde{u}_{ij} + 2a^{1+\alpha}\tilde{\Pi}^{\mathrm{eff}}_{ij} \; , \end{cases} \tag{16}$$

which can be solved using finite difference methods, see Ref. [6] for a description of the available algorithms in $\mathcal{C}$osmo$\mathcal{L}$attice. The gravitational wave energy density power spectrum normalized by the critical energy can then be computed in different ways, see Ref. [1, 4] for a discussion of the options available in $\mathcal{C}$osmo$\mathcal{L}$attice. By default, the Type I - Version 1 power spectrum is computed, in which the exact *multiplicity* of modes per bin, $\#_l$, is counted and all modes are assigned the central k value for the bin,

$$\Omega_{\mathrm{GW}}(\tilde{\mathbf{n}}, t) = \frac{1}{\rho_{\mathrm{c}}} \frac{m_{\mathrm{p}}^2 k(l)}{8\pi} \frac{\delta x}{N^5} \#_l \left\langle \left[\dot{h}_{ij}(\tilde{\mathbf{n}}, t)\right] \left[\dot{h}^*_{ij}(\tilde{\mathbf{n}}, t)\right] \right\rangle_{R(l)} \; , \tag{17}$$

where the different variables are defined in Ref. [1]. In program variables this is expressed as

$$\Omega_{\mathrm{GW}}(\tilde{\mathbf{n}}, t) = \frac{1}{\tilde{\rho}_{\mathrm{c}}} \frac{\kappa(l)}{8\pi a^{2\alpha}} \left(\frac{\delta\tilde{x}}{N^5}\right) \left(\frac{f_*}{m_{\mathrm{p}}}\right)^2 \#_l a^{-2(3-\alpha)} \langle [\Lambda_{ij,kl}(\tilde{\mathbf{n}})(\pi_{\tilde{u}})_{kl}(\tilde{\mathbf{n}}, t)][\Lambda_{ij,mn}(\tilde{\mathbf{n}})(\pi_{\tilde{u}})_{mn}(\tilde{\mathbf{n}}, t)]^* \rangle_{R(l)} \; . \tag{18}$$

For completeness, we also quote the Type II - Version 1 power spectrum, in which the number of modes per bin is approximated $\#_l \approx 4\pi l^2$. In this case, the normalized GW energy density power spectrum is defined as

$$\Omega_{\mathrm{GW}}(\tilde{\mathbf{n}}, t) = \frac{1}{\rho_{\mathrm{c}}} \frac{m_{\mathrm{p}}^2 k^3(l)}{8\pi^2} \frac{\delta x^3}{N^3} \left\langle \left[\dot{h}_{ij}(\tilde{\mathbf{n}}, t)\right] \left[\dot{h}^*_{ij}(\tilde{\mathbf{n}}, t)\right] \right\rangle_{R(l)} \; , \tag{19}$$

and is expressed in program variables as

$$\Omega_{\mathrm{GW}}(\tilde{\mathbf{n}}, t) = \frac{1}{\tilde{\rho}_{\mathrm{c}}} \frac{\kappa^3(l)}{(8\pi^2 a^{2\alpha})} \left(\frac{\delta\tilde{x}}{N}\right)^3 \left(\frac{f_*}{m_{\mathrm{p}}}\right)^2 a^{-2(3-\alpha)} \langle [\Lambda_{ij,kl}(\tilde{\mathbf{n}})(\pi_{\tilde{u}})_{kl}(\tilde{\mathbf{n}}, t)][\Lambda_{ij,mn}(\tilde{\mathbf{n}})(\pi_{\tilde{u}})_{mn}(\tilde{\mathbf{n}}, t)]^* \rangle_{R(l)} \; . \tag{20}$$

Important Note -. While the GW energy density spectrum at the time of production Ω_{GW} is typically normalized in an expanding universe by the critical energy density $\tilde{\rho}_{\mathrm{c}}$, in $\mathcal{C}$osmo$\mathcal{L}$attice we rather normalize it by the total energy density of the matter field sector $\tilde{\rho}_{\mathrm{tot}}$ (let it be composed of scalar fields only, or scalar and gauge fields), independently of whether we simulate the dynamics in an expanding background or in Minkowski. In the case of self-consistent expansion $\tilde{\rho}_{\mathrm{tot}} = \tilde{\rho}_{\mathrm{c}}$, and hence we recover the standard definition. However, for a fixed-background expansion, if the user wishes to obtain the spectrum normalized to the critical energy density, they should multiply the $\mathcal{C}$osmo$\mathcal{L}$attice output (second column of `spectra_gws.txt`) by the ratio $\tilde{\rho}_{\mathrm{tot}}/\tilde{\rho}_{\mathrm{c}}$.

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