CosmoLattice

School 2022 - Valencia

Non-linear dynamics of Axion-inflation

Joanes Lizarraga & Ander Urio

Basics of the model

[K. Freese, J. A. Frieman, A. V. Olinto (PRL 65,3233 1990)]
...

$$\boxed{\text{Axion-inflation} \quad + \quad \frac{\phi}{\Lambda} F\tilde{F}}$$

Shift symmetry $\phi \to \phi + c$

Action:

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} m_{\rm pl}^2 R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\phi}{4\Lambda} F_{\mu\nu} \tilde{F}^{\mu\nu} \right)$$

$\phi \longrightarrow$ Pseudo-scalar axion field

[M. M. Anber, L. Sorbo (0908.4089)]
[J. Cook, L. Sorbo (1109.0022)]
[N. Barnaby, E. Pajer, M. Peloso (1110.3327)]

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$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

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Simple derivative
Non covariant

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Generated from external mechanism breaks the shift symmetry explicitly

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Interaction only through this vertex
Axion coupling!

$\phi \longrightarrow$ Pseudo-scalar axion field

$1/\Lambda$ Coupling constant $[m_{\rm pl}^{-1}]$

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Vectorial form

$$S = \int d^4x \left[\frac{1}{2} a^3 \pi_\phi^2 - \frac{1}{2} a \left(\vec{\nabla}\phi\right)^2 - \frac{1}{2} a^3 m^2 \phi^2 + \frac{1}{2} a \left(\vec{E}^2 - \frac{\vec{B}^2}{a^2} \right) + \frac{\phi}{\Lambda} \vec{E}\cdot\vec{B} \right]$$

@ FLRW:

$ds^2 = -dt^2 + a^2 d\vec{x}^2$

In cosmic time

Basics of the model

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In cosmic time

Our choice of the potential, also:

$$V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{2}{3}}\phi} \right)^2$$

$$V(\phi) = V_0 \left(1 - \left(\frac{\phi}{\eta} \right)^4 \right)^2$$

$$V(\phi) = \ldots$$

3 vertex

Basics of the model

Continuum equations of motion:

Temporal gauge

$$A_0 = 0$$

CL momentum variables

$$\tilde{\pi}_\phi = a^3\pi_\phi, \quad \tilde{\vec{E}} = a\vec{E}$$

No Hubble friction term in the EoMs

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Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2 \phi + \frac{1}{a\Lambda}\tilde{\vec{E}} \cdot \vec{B} \ ,$$

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Constraint equation: Gauss's law

$$\vec{\nabla}\cdot\tilde{\vec{E}} + \frac{1}{\Lambda}\vec{\nabla}\phi\cdot\vec{B} = 0$$

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Non-linear terms

Constraint equation:
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Basics of the model: linear regime

Let's go now deep inside inflation

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Inflaton in slow-roll,
Homogeneous
+
dominated by the potential

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- No backreaction in inflation's dynamics
- No generation of inhomogeneities

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Gauge fields get excited by this term

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EoMs

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Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) = \frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

$$\left(\begin{array}{c} K_\phi \equiv \frac{1}{2}\pi_\phi^2 = \frac{1}{2a^6}\tilde{\pi}_\phi^2\,, \quad G_\phi \equiv \frac{1}{2a^2}\sum(\partial_i\phi)^2\,, \quad V \equiv \frac{1}{2}m^2\phi^2\,, \\ K_A \equiv \frac{1}{2}\sum_i \frac{E_i^2}{a^2} = \frac{1}{2}\sum_i\frac{\tilde{E}_i^2}{a^4}\,, \quad G_A \equiv \frac{1}{2}\sum_i\frac{B_i^2}{a^4} \end{array}\right)$$

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Gauge fields evolve in the background set by the axion field

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Basics of the model: linear regime

$\boxed{\text{An analitycal solution can be found}}$

Changing to the helicity basis: + Conformal time

$$\vec{A}(\vec{x},\tau) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} A^{\lambda}(k,\tau)\vec{\varepsilon}^{\lambda}(\hat{k})e^{i\vec{k}\cdot\vec{x}}$$

Photon with 2 helicity states

$$k_i\varepsilon_i^{\pm}(\hat{k}) = 0, \quad \epsilon_{ijk}k_j\varepsilon_k^{\pm}(\hat{k}) = \mp ik\varepsilon_i^{\pm}(\hat{k})$$

$$\varepsilon_i^{\pm}(\hat{k})^* = \varepsilon_i^{\pm}(-\hat{k}), \quad \varepsilon_i^{\lambda}(\hat{k})\varepsilon_i^{\lambda'}(\hat{k}) = \delta_{\lambda,\lambda'}$$

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Two polarisations:
- One exponentially amplified
- The other not

Chiral instability

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Changing to the helicity basis: + Conformal time

$$\vec{A}(\vec{x},\tau) = \sum_{\lambda=\pm}\int \frac{d^3k}{(2\pi)^3} A^\lambda(k,\tau)\vec{\varepsilon}^\lambda(\hat{k})e^{i\vec{k}\cdot\vec{x}}$$

Photon with 2 helicity states

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

$$\xi \equiv \frac{\dot{\phi}}{2H\Lambda}$$

$$k_i\varepsilon_i^\pm(\hat{k}) = 0, \quad \epsilon_{ijk}k_j\varepsilon_k^\pm(\hat{k}) = \mp ik\varepsilon_i^\pm(\hat{k})$$

$$\varepsilon_i^\pm(\hat{k})^* = \varepsilon_i^\pm(-\hat{k}), \quad \varepsilon_i^\lambda(\hat{k})\varepsilon_i^{\lambda'}(\hat{k}) = \delta_{\lambda,\lambda'}$$

Two polarisations:
- One exponentially amplified
- The other not

Chiral instability

$$|A^+| \gg |A^-|$$

Basics of the model: linear regime

Analytical solution:

[M. M. Anber, L. Sorbo (0908.4089)]

$$\boxed{\xi \equiv \frac{\dot{\phi}}{2H\Lambda}}$$

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Deep inside inflation $\xi \to \mathrm{const}$

$$A^+(k,t) = \frac{e^{\frac{\pi}{2}\xi}}{\sqrt{2k}} W_{i\xi,\frac{1}{2}}(2ikt) \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{2|\xi|aH}\right)^{1/4} e^{\pi|\xi| - 2\sqrt{2|\xi|k/aH}}$$

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Topological term:

Appears in EoMs

$$\frac{1}{a^3}\langle \vec{E}\cdot\vec{B}\rangle \simeq 2.4\cdot 10^{-4}\frac{H^4}{|\xi|^4}e^{2\pi|\xi|}$$

EM energy:

$$\frac{1}{2a^4}\langle a^2E^2 + B^2\rangle \simeq 1.4\cdot 10^{-4}\frac{H^4}{|\xi|^3}e^{2\pi|\xi|}$$

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Amplified around the Hubble scale at each time

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Instability controlled by ξ

Deep inside inflation $\xi \to \mathrm{const}$

Not accurate towards the end of inflation

$$A^+(k,t) = \frac{e^{\frac{\pi}{2}\xi}}{\sqrt{2k}} W_{i\xi,\frac{1}{2}}(2ikt) \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{2|\xi|aH}\right)^{1/4} e^{\pi|\xi| - 2\sqrt{2|\xi|k/aH}}$$

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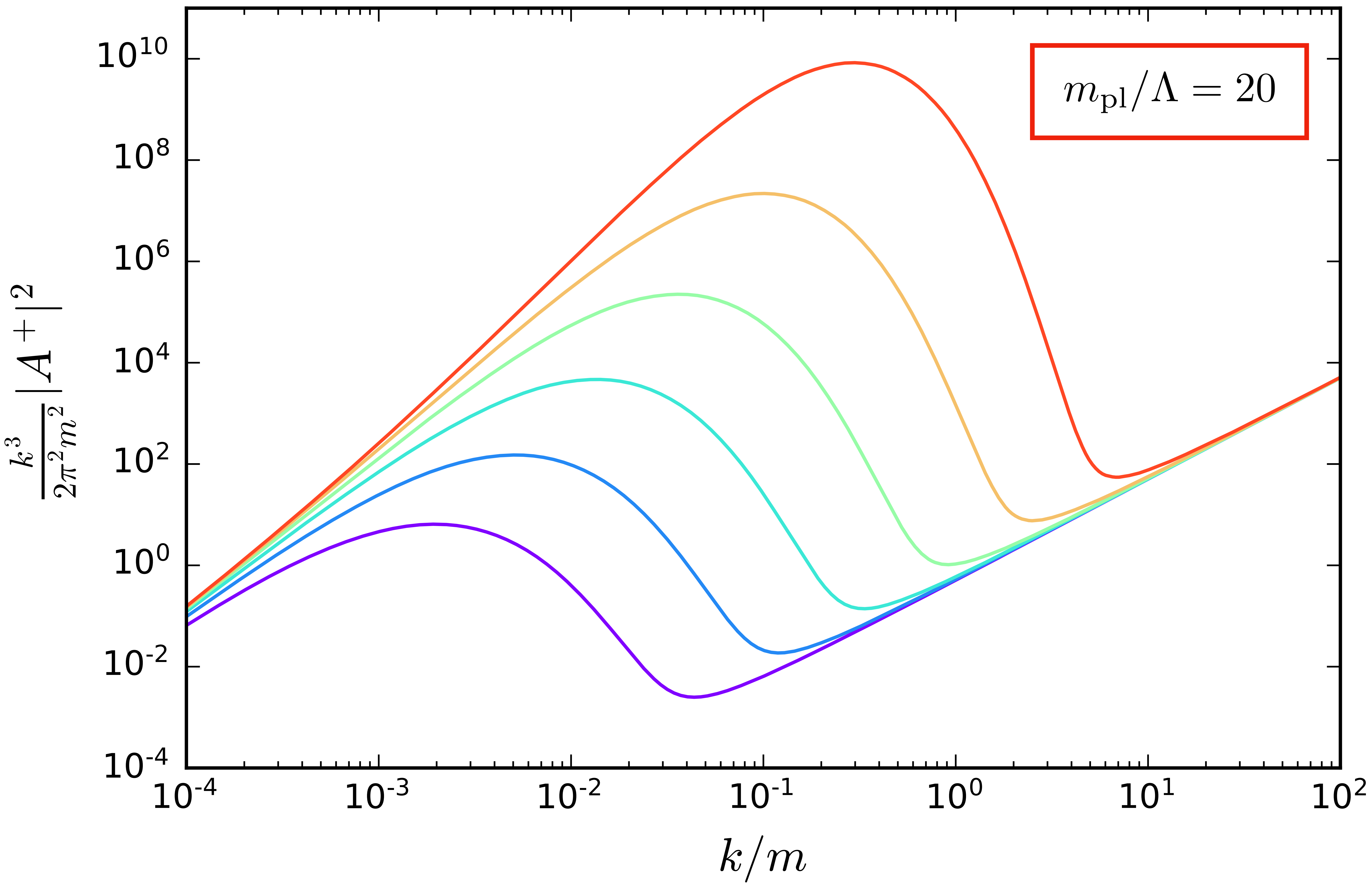
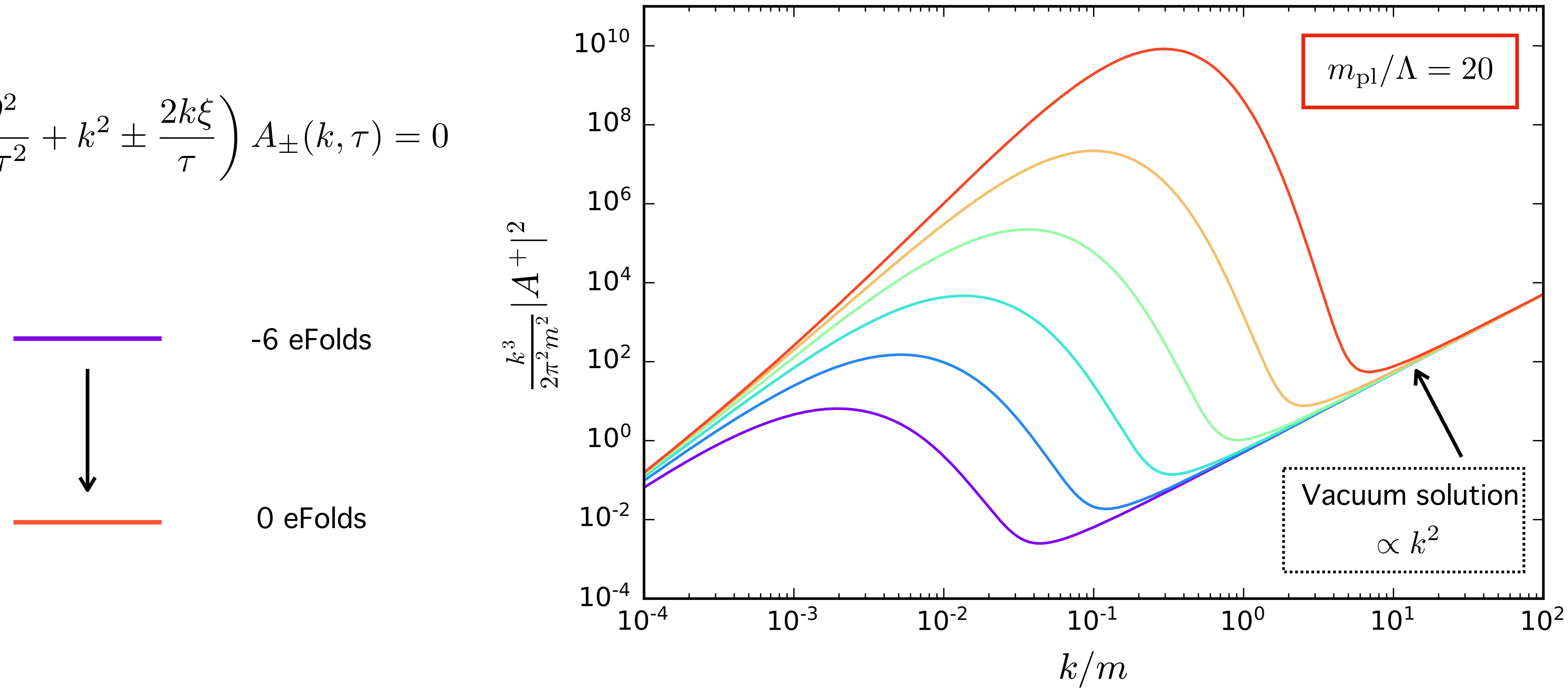
-6 eFolds

0 eFolds



Basics of the model: linear regime

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$



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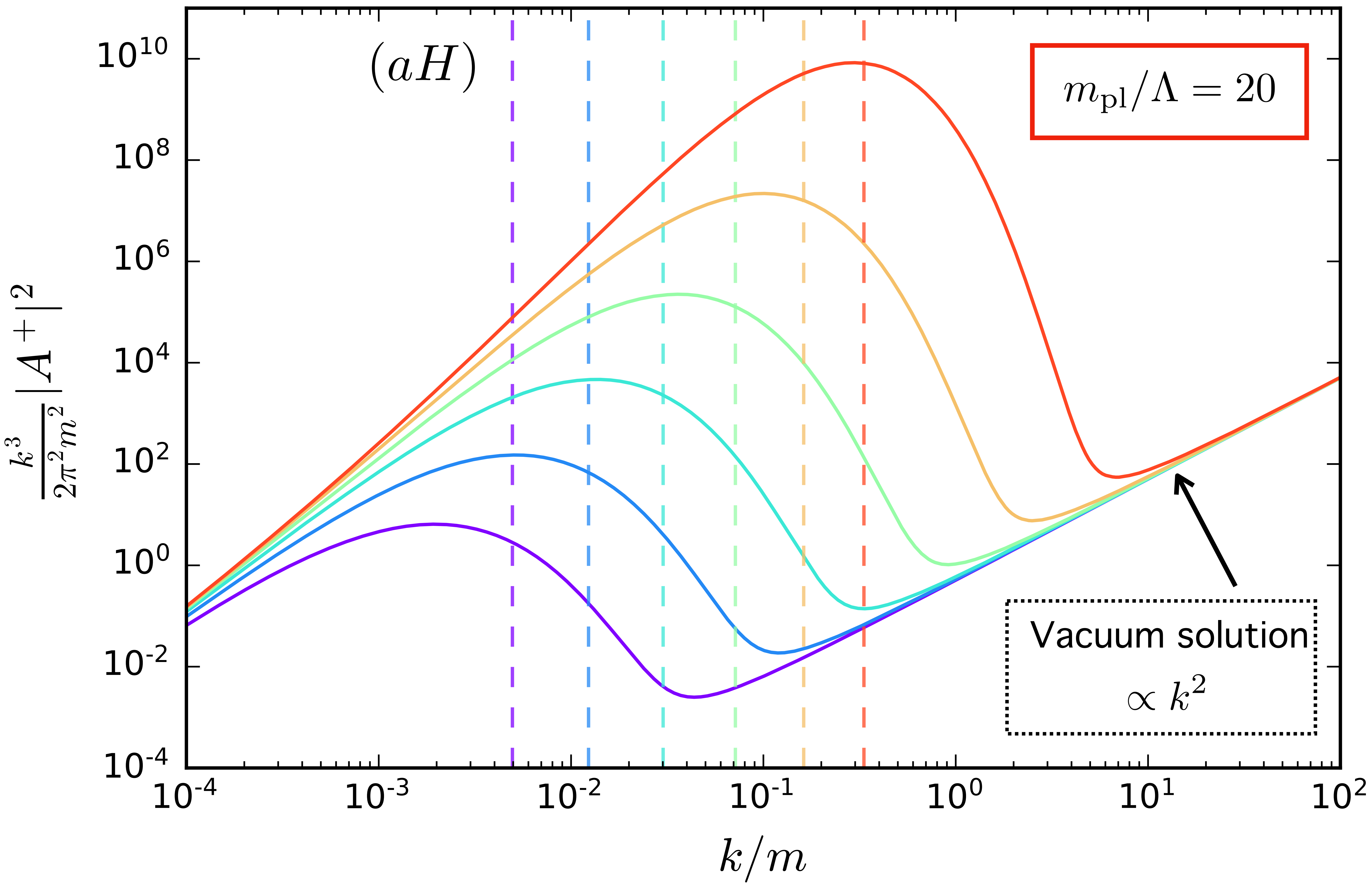
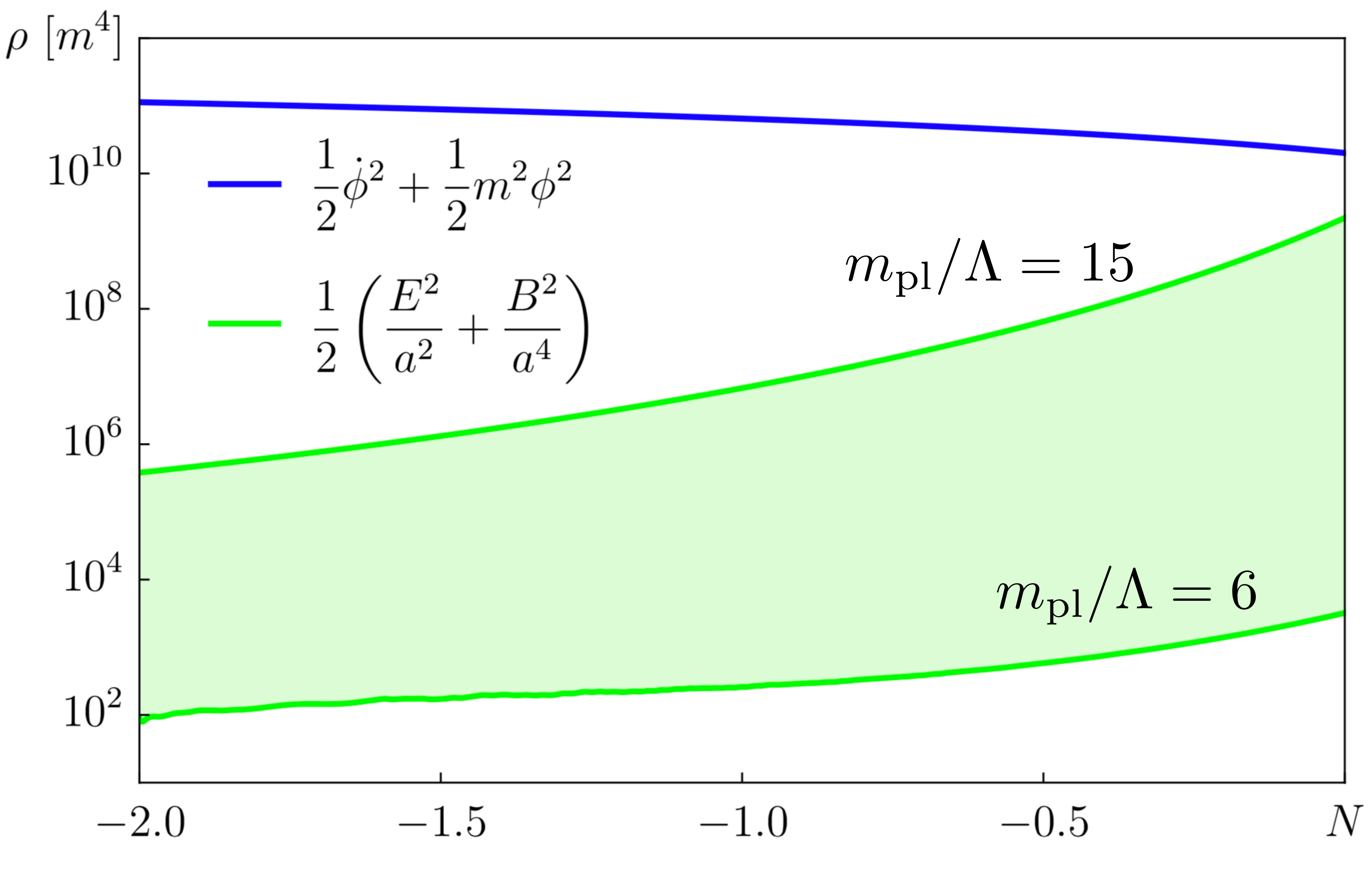


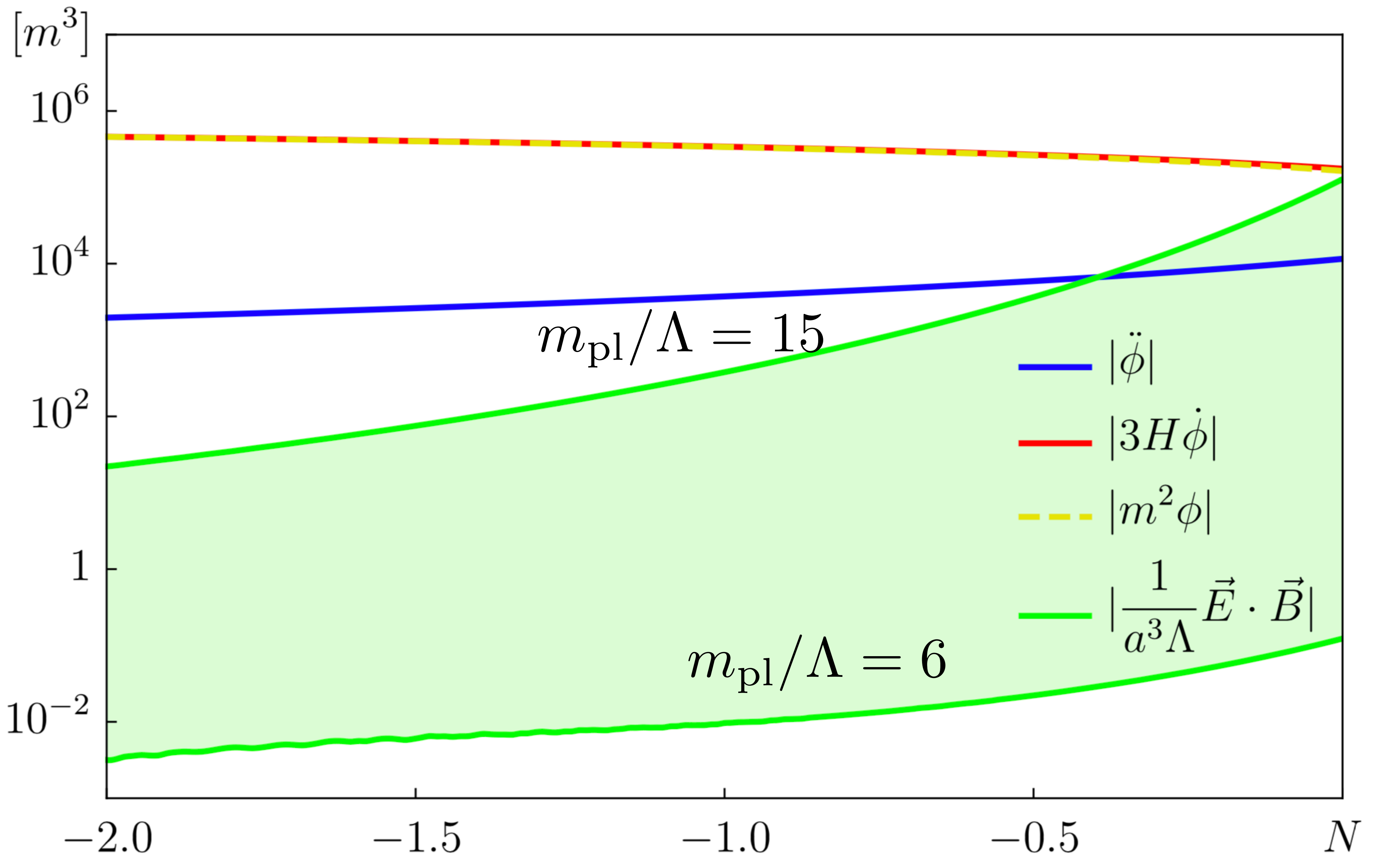
Basics of the model: linear regime

[J. R. C. Cuissa, D. G. Figueroa (1812.03132)]

Energy components



Topological term vs slow-roll



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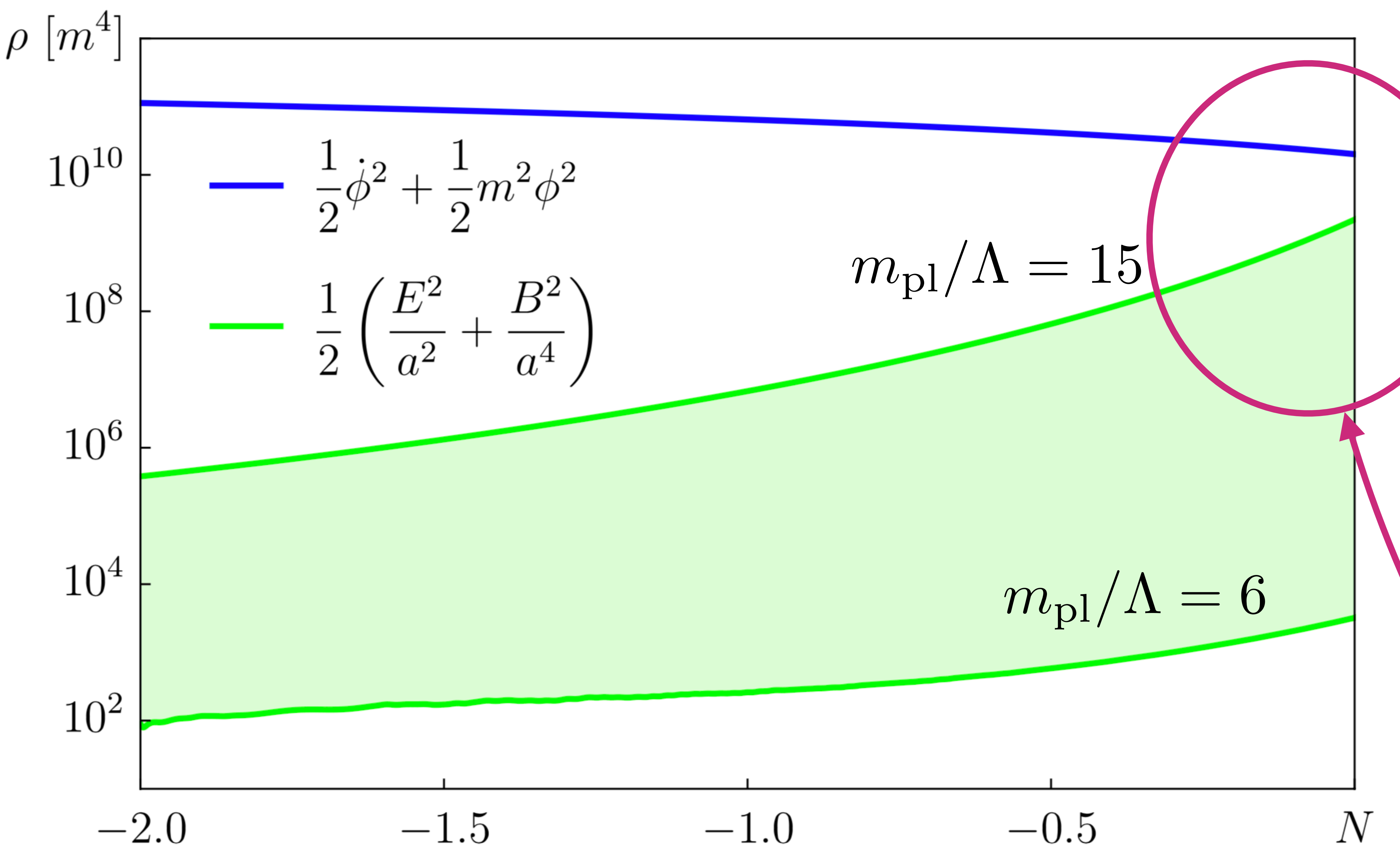

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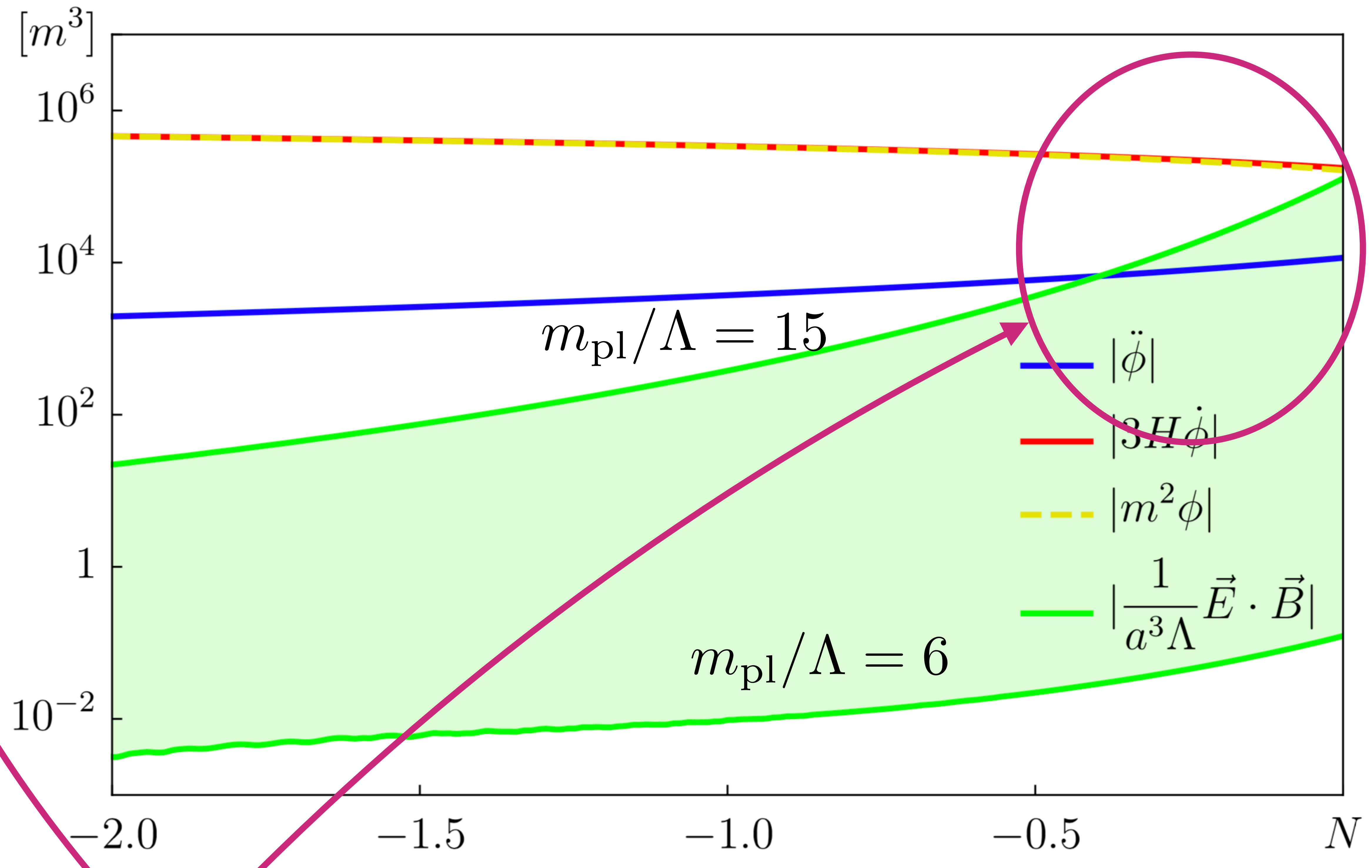

Backreaction effect cannot be neglected

Basics of the model: linear regime

Linear regime

Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\,,$$

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Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) = \frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

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Neglected

Basics of the model

Beyond the linear regime: backreaction

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Backreaction terms

Beyond the linear regime: backreaction

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+
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Basics of the model

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Dynamical equations:

1st approximation:
homogeneous

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Due to backreaction not only slow-roll

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Why in the lattice?

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No gradient terms induced (yet)

Expansion equation:

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Lattice provides the solution: Real time evolution of fields

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Inhomogeneous source:
Induce gradients not
tractable analytically

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tractable analytically

Why in the lattice?

Lattice provides the solution:
Real time evolution of fields

Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\,,$$

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}$$

Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) = \frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A\rangle$$

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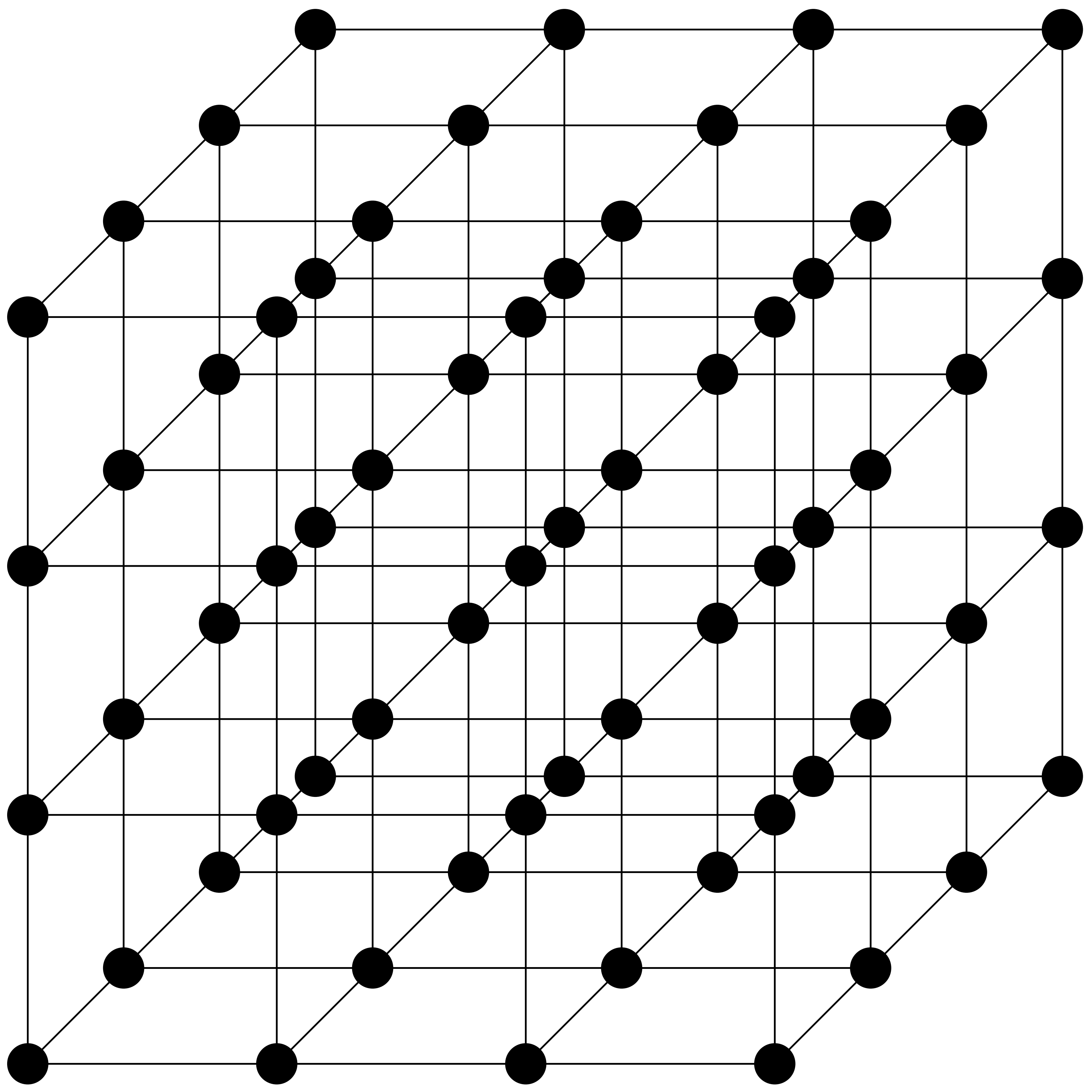
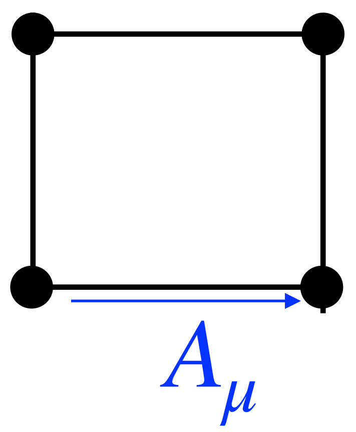
$\forall$ D.o.F will be simultaneously evolved

- Track of all inhomogeneities
- Full backreaction effects included

Discretisation procedure





ϕ @ sites

A_μ @ links

[D. G. Figueroa & M. Shaposhnikov (1705.09629)]

Want to preserve:

1- Gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

2- Bianchi identities

$$\vec{\nabla} \times \vec{E} = \dot{\vec{B}} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

3- Topological nature of the coupling

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$$

Exact lattice shift symmetry

4- Continuum limit to $\mathcal{O}(dx^2)$

Discretisation procedure

[D. G. Figueroa, M. Shaposhnikov (1705.09629)]

Continuum

$$\frac{\phi}{\Lambda}\vec{E}\cdot\vec{B}$$

The only one that satisfies all conditions

Lattice

$$\boxed{\sum_i \frac{\phi}{\Lambda} E_i^{(2)} B_i^{(4)}}$$

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$$E_i^{(2)} \equiv \frac{1}{2}(E_i + E_{i,-i}) \quad @ \quad n + \frac{\hat{0}}{2}$$

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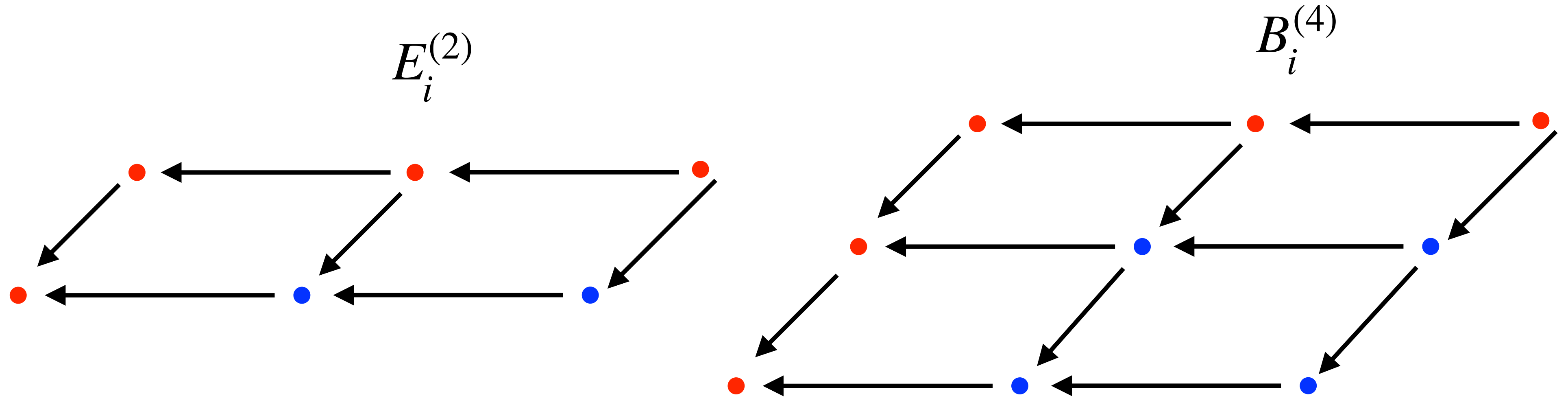

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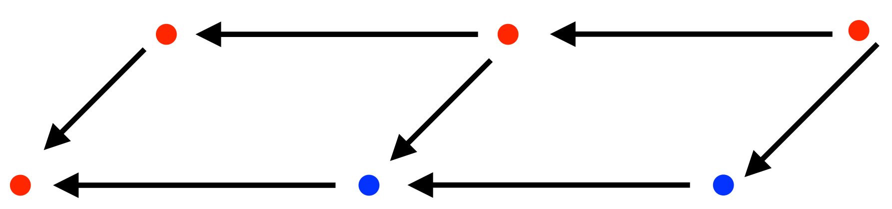

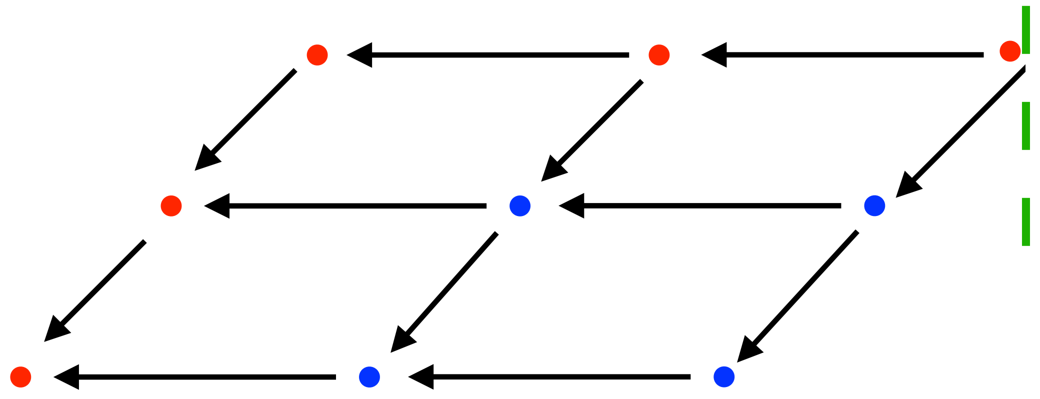

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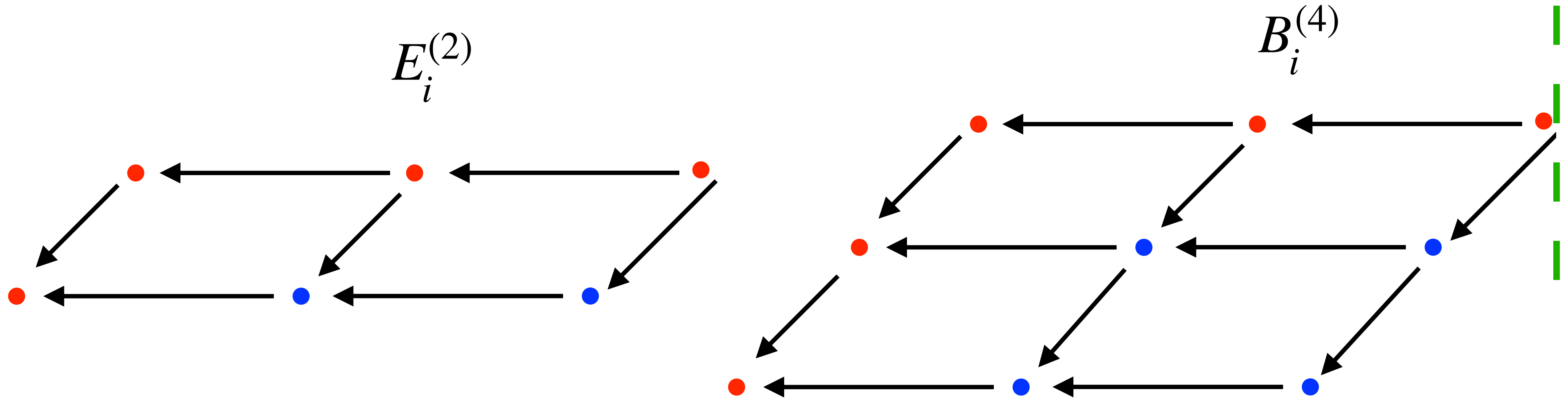

Discretisation procedure

[D. G. Figueroa, M. Shaposhnikov (1705.09629)]
[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]

$$S = \int d^4x \left[\frac{1}{2} a^3 \pi_\phi^2 - \frac{1}{2} a \left(\vec{\nabla}\phi \right)^2 - \frac{1}{2} a^3 m^2 \phi^2 + \frac{1}{2} a \left(\vec{E}^2 - \frac{\vec{B}^2}{a^2} \right) + \frac{\phi}{\Lambda} \vec{E} \cdot \vec{B} \right]$$

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Full discretised action:

$$S = \Delta t\Delta x^3 \sum_{t,\vec{n}} \left[\frac{1}{2a^3}\left(\tilde{\pi}_\phi\right)^2 - \frac{1}{2}a_{+\frac{\hat{0}}{2}}\left(\Delta_i^+\phi_{+\frac{\hat{0}}{2}}\right)^2 - \frac{1}{2}a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}}\right.$$
$$+ \frac{1}{2}a_{+\frac{\hat{0}}{2}}\sum_i\left(\Delta_0^+A_i - \Delta_i^+A_0\right)^2 - \frac{1}{4a}\sum_{i,j}\left(\Delta_i^+A_j - \Delta_j^+A_i\right)^2$$
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Discretisation procedure

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$$+ \frac{1}{2} a_{+\frac{\hat{0}}{2}} \sum_i \left(\Delta_0^+ A_i - \Delta_i^+ A_0 \right)^2 - \frac{1}{4a} \sum_{i,j} \left(\Delta_i^+ A_j - \Delta_j^+ A_i \right)^2$$

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Full discretised action:

Ready for variational principle

$$S = \Delta t \Delta x^3 \sum_{t,\vec{n}} \Bigg[\frac{1}{2a^3}\left(\cdots\right. \cdots \frac{}{2}a_{+\frac{\hat{0}}{2}}\left(\Delta_i^+\phi_{+\frac{\hat{0}}{2}}\right)^2 - \frac{1}{2}a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}}$$

$$+ \frac{1}{2}a_{+\frac{\hat{0}}{2}}\sum_i\left(\Delta_0^+A_i - \Delta_i^+A_0\right)^2 - \frac{1}{4a}\sum_{i,j}\left(\Delta_i^+A_j - \Delta_j^+A_i\right)^2$$

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Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^L_{\phi} \ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_{\phi} \ , \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

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Dynamical equations:

$$\mathcal{K}_{\phi}^{L} = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)} \right)$$

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Gauss's constraint:

$$\sum_i \Delta_i^- \tilde{E}_{i,+\frac{\hat{0}}{2}} = -\frac{1}{4\Lambda} \sum_\pm \sum_i \left(\Delta_i^\pm \phi_{+\frac{\hat{0}}{2}} \right) \left(B_i^{(4)} + B_{i,+\hat{0}}^{(4)} \right)_{\pm i}$$

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Common kick-drift scheme

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Common kick-drift scheme

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Alternative approach in:
[A. Caravano, E. Komatsu, K. D. Lozanov, J. Weller (2204.12874)]

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$b_{+\hat{0}} = b + dt\mathcal{K}_a^L$$

$$a_{+\frac{\hat{0}}{2}} = a_{-\frac{\hat{0}}{2}} + dtb$$

Expansion equations:

$$\boxed{\mathcal{K}_a^L = -\frac{a_{+\hat{0}/2}}{6m_{\rm pl}^2}(\rho_L + 3p_L)_{+\hat{0}/2}}$$

Hubble's constraint:

$$\boxed{\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L}$$

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$b_{+\hat{0}} = b + dt\mathcal{K}_a^L$$

$$a_{+\frac{\hat{0}}{2}} = a_{-\frac{\hat{0}}{2}} + dtb$$

Expansion equations:

$$\boxed{\mathcal{K}_a^L = -\frac{a_{+\hat{0}/2}}{6m_{\rm pl}^2}(\rho_L + 3p_L)_{+\hat{0}/2}}$$

$$(\rho_L + 3p_L)_{+\hat{0}/2} = 2(\bar{K}_\phi^L + \bar{K}_{\phi,+\hat{0}}^L) - 2\bar{V}_{\phi,+\hat{0}/2}^L + 2K_A^L + (G_A^L + G_{A,+\hat{0}}^L)$$

$$\rho_L = \bar{K}_\phi^L + \frac{1}{2}(\bar{G}_{\phi,-\hat{0}/2}^L + \bar{G}_{\phi,+\hat{0}/2}^L) + \frac{1}{2}(\bar{V}_{\phi,-\hat{0}/2}^L + \bar{V}_{\phi,+\hat{0}/2}^L) + \frac{1}{2}(\bar{K}_{A,-\hat{0}/2}^L + \bar{K}_{A,+\hat{0}/2}^L) + \bar{G}_A^L$$

Hubble's constraint:

$$\boxed{\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L}$$

$$\bar{K}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2a^6}\tilde{\pi}_\phi^2, \quad \bar{G}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}\sum_i(\Delta_i^+\phi_{+\hat{0}/2})^2, \quad \bar{V}_\phi^L = \frac{1}{N^3}\sum_{\vec{n}}\frac{1}{2}m^2\phi_{+\hat{0}/2}^2$$

$$\bar{K}_A^L = \frac{1}{N^3}\sum_{\vec{n}}\sum_i\frac{1}{2a_{+\hat{0}/2}^4}\tilde{E}_{i,+\hat{0}/2}^2, \quad \bar{G}_A^L = \frac{1}{N^3}\sum_{\vec{n}}\sum_i\frac{1}{4a^4}(\Delta_i^+A_j - \Delta_j^+A_i)^2$$

Discretisation procedure

Evolution Kernels

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda} (2 + dx\Delta_i^+) \sum_\pm \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda} (2 + dx\Delta_i^+) \sum_\pm \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Non-symplectic,
Conjugate momenta in kernels!
[Remember 4th lecture & 1st Topical]

$$\dot{\tilde{E}}_i = \mathcal{K}^L_{i,A} \left[a, b, \phi, A_i, \tilde{E}_i \right]$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda} \sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}} \right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta_j^- B_k - \frac{1}{2a^3\Lambda} \left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \right)$$
$$+ \frac{1}{8a\Lambda} (2 + dx \Delta_i^+) \sum_\pm \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi) \tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Non-symplectic,
Conjugate momenta in kernels!
[Remember 4th lecture & 1st Topical]

$$\dot{\tilde{E}}_i = \mathcal{K}^L_{i,A} \left[a, b, \phi, A_i, \tilde{E}_i \right]$$

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^{L}_{\phi} \ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^{L}_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_{\phi} \ , \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_\phi + dt\mathcal{K}^L_\phi \ , \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_\phi \ , \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda}\sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}} \left(B_i^{(4)} + B^{(4)}_{i,+\hat{0}}\right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i}\right)$$
$$+\frac{1}{8a\Lambda}(2 + dx\Delta_i^+)\sum_{\pm}\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

Discretisation procedure

Common kick-drift scheme

Evolution Kernels

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_\phi + dt\mathcal{K}^L_\phi\,, \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{a^3}\tilde{\pi}_\phi\,, \quad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Dynamical equations:

$$\mathcal{K}^L_\phi = a_{+\frac{\hat{0}}{2}}\sum_i \Delta^-_i\Delta^+_i\phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}}m^2\phi_{+\frac{\hat{0}}{2}} + \frac{1}{2a\Lambda}\sum_i \tilde{E}^{(2)}_{i,+\frac{\hat{0}}{2}}\left(B^{(4)}_i + B^{(4)}_{i,+\hat{0}}\right)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta^-_j B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B^{(4)}_i + \tilde{\pi}_{\phi,+i}B^{(4)}_{i,+i}\right)$$
$$+\frac{1}{8a\Lambda}(2 + dx\Delta^+_i)\sum_\pm\sum_{j,k}\left\{\epsilon_{ijk}[(\Delta^\pm_j\phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta^\pm_j\phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}}\right\}$$

Implicit scheme required

We need the value of the gauge's canonical momentum @ 0/2 for obtaining its value @ 0/2!

Discretisation procedure

Common kick-drift scheme

$$\tilde{\pi}_{\phi,+\hat{0}} = \tilde{\pi}_{\phi} + dt\mathcal{K}^L_{\phi}\,, \quad \tilde{E}_{i,+\frac{\hat{0}}{2}} = \tilde{E}_{i,-\frac{\hat{0}}{2}} + dt\mathcal{K}^L_{i,A}$$

$$\phi_{+\frac{\hat{0}}{2}} = \phi_{-\frac{\hat{0}}{2}} + \frac{dt}{\ldots} \ldots \qquad A_{i,+\hat{0}} = A_i + \frac{dt}{a}\tilde{E}_{i,+\frac{\hat{0}}{2}}$$

Evolution Kernels

Dynamical equations:

$$\mathcal{K}^L_{\phi} = a_{+\frac{\hat{0}}{2}} \sum_i \Delta_i^- \Delta_i^+ \phi_{+\frac{\hat{0}}{2}} - a^3_{+\frac{\hat{0}}{2}} m^2 \phi \ldots + B^{(4)}_{i,+\hat{0}} \Big)$$

$$\mathcal{K}^L_{i,A} = -\frac{1}{a} \sum_{j,k} \epsilon_{ijk} \Delta^- \ldots + \tilde{\pi}_{\phi,+i} B^{(4)}_{i,+i} \Big)$$

$$+ \frac{1}{8a} \ldots \sum_{\pm} \sum_{j,k} \left\{ \epsilon_{ijk} [(\Delta_j^{\pm}\phi)\tilde{E}^{(2)}_{k,\pm j}]_{+\frac{\hat{0}}{2}} + [(\Delta_j^{\pm}\phi)\tilde{E}^{(2)}_{k,\pm j}]_{-\frac{\hat{0}}{2}} \right\}$$

Need for a non-symplectic integrator!
[Ander]

Implicit scheme required

We need the value of the gauge's
canonical momentum @ 0/2 for
obtaining its value @ 0/2!

ANDER URIO

Our Current Work

Explicit-in-time integrator

Joanes

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Ander

Explicit-in-time integrator

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If momenta appear in kernels

Not solvable by explicit symplectic integrators

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Explicit-in-time integrator

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If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

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If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

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Explicit-in-time integrator

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If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

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Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Runge-Kutta

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

Runge-Kutta

(explicit version)

Explicit-in-time integrator

Joanes

If momenta appear in kernels

Not solvable by explicit symplectic integrators

i.e. LF or VV

Why?

$$\pi_{+\frac{\hat{0}}{2}} = \pi_{-\frac{\hat{0}}{2}} + \mathcal{K}[\pi_{+\frac{\hat{0}}{2}}]dt$$

Solution

Ander

Non-sympletic integrators

(explicit version)

Runge-Kutta

No longer!

Explicit-in-time integrator

$\boxed{1^{\text{st}} \text{ order ODE}}$

$$\frac{dx}{dt} = v(t, x)$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t, x)$$

Explicit-in-time integrator

RK2 & 2 stages

1 st order ODE

$$\frac{dx}{dt} = v(t,x) \quad \Rightarrow \quad \left\{ \; x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \; \right\}$$

Explicit-in-time integrator

RK2 & 2 stages

1 st order ODE

$$\frac{dx}{dt} = v(t, x) \Rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t,x) \Rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\mathrm{Cont}}(t_{n+1}) - x^{\mathrm{RK2}}(t_{n+1}) = \mathcal{O}(dt^3)$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t,x) \Rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\mathrm{Cont}}(t_{n+1}) - x^{\mathrm{RK2}}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE $$\frac{dx}{dt} = v(t,x)$$ $\Rightarrow$

$$\left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\rm Cont}(t_{n+1}) - x^{\rm RK2}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

Modified Euler

Explicit-in-time integrator

$\boxed{\text{RK2 \& 2 stages}}$

$\boxed{1^{\rm st}\text{ order ODE}}$

$$\frac{dx}{dt} = v(t,x) \;\Rightarrow\; \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\rm Cont}(t_{n+1}) - x^{\rm RK2}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\,,\;\; b_2 c_2 = 1/2\,,\;\; b_2 a_{21} = 1/2$$

$\boxed{\text{Modified Euler}}$

$$b_1 = 1/2\,,\;\; b_2 = 1/2\,,\;\; c_2 = 1\,, a_{21} = 1$$

Explicit-in-time integrator

RK2 & 2 stages

1st order ODE

$$\frac{dx}{dt} = v(t,x) \rightarrow \left\{ \begin{array}{c} x(t_{n+1}) = x(t_n) + dt\,(b_1 G_1 + b_2 G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + c_2 dt, x(t_n) + a_{21} G_1 dt) \end{array} \right\}$$

$$x^{\mathrm{Cont}}(t_{n+1}) - x^{\mathrm{RK2}}(t_{n+1}) = \mathcal{O}(dt^3)$$

$$\text{Constraints} \rightarrow b_1 + b_2 = 1\ ,\ \ b_2 c_2 = 1/2\ ,\ \ b_2 a_{21} = 1/2$$

Modified Euler

$$b_1 = 1/2\ ,\ \ b_2 = 1/2\ ,\ \ c_2 = 1\ , a_{21} = 1$$

$$\boxed{\begin{array}{c} x(t_{n+1}) = x(t_n) + \dfrac{dt}{2}\,(G_1 + G_2) \\ G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + dt, x(t_n) + G_1 dt) \end{array}}$$

Explicit-in-time integrator

Explicit-in-time integrator

$\boxed{\text{2}^{\text{nd}}\text{ order ODE}}$

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Explicit-in-time integrator

$\boxed{\text{2}^{\text{nd}}\text{ order ODE}}$

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Explicit-in-time integrator

2nd order ODE

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Coupled 1st order ODEs

1st order ODE RK2 ?

Explicit-in-time integrator

2nd order ODE

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = v(t,x) \qquad \longleftarrow \qquad \text{Coupled 1st order ODEs} \qquad \longrightarrow \qquad \frac{dv}{dt} = \mathcal{K}(t,x,v)$$

Explicit-in-time integrator

2nd order ODE

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Coupled 1st order ODEs

$$\frac{dx}{dt} = v(t,x)$$

$$\frac{dv}{dt} = \mathcal{K}(t,x,v)$$

RK

$$\begin{gathered} G_1 = v(t_n, x(t_n)) \\ G_2 = v(t_n + dt, x(t_n) + G_1 dt) \\ x(t_{n+1}) = x(t_n) + \frac{dt}{2}(G_1 + G_2) \end{gathered}$$

Explicit-in-time integrator

2nd order ODE

1st order ODE RK2 ?

$$\frac{d^2x}{dt^2} = \mathcal{K}\left(t, x, \frac{dx}{dt}\right)$$

Coupled 1st order ODEs

$$\frac{dx}{dt} = v(t,x)$$

$$\frac{dv}{dt} = \mathcal{K}(t,x,v)$$

RK

$$\begin{aligned} G_1 &= v(t_n, x(t_n)) \\ G_2 &= v(t_n + dt, x(t_n) + G_1 dt) \\ x(t_{n+1}) &= x(t_n) + \frac{dt}{2}(G_1 + G_2) \end{aligned}$$

RK

$$\begin{aligned} G_1^* &= \mathcal{K}(t_n, x(t_n), v(t_n)) \\ G_2^* &= \mathcal{K}(t_n + dt, x(t_n) + dtG_1, v(t_n) + dtG_1^*) \\ v(t_{n+1}) &= v(t_n) + \frac{dt}{2}(G_1^* + G_2^*) \end{aligned}$$

Explicit-in-time integrator

Axion-Inflation using RK2

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1) \quad \begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases} \longrightarrow \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \longrightarrow \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi \left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A \left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a \left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right] \ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases} \qquad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \qquad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right] \ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases} \rightarrow \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \rightarrow \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_n \equiv (1)$$

$$\begin{cases} \phi^{(1)} = \phi(t_n) \ , \\ A_i^{(1)} = A_i(t_n) \ , \\ a^{(1)} = a(t_n) \ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n) \ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n) \ , \\ \pi_a^{(1)} = \pi_a(t_n) \ , \end{cases} \quad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right] \ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right] \ . \end{cases}$$

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases} \quad \begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases} \quad \begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \tilde{\vec{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases} \quad \begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases} \quad \begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_n \equiv (1)$

$$\begin{cases} \phi^{(1)} = \phi(t_n)\ , \\ A_i^{(1)} = A_i(t_n)\ , \\ a^{(1)} = a(t_n)\ , \\ \tilde{\pi}_\phi^{(1)} = \tilde{\pi}_\phi(t_n)\ , \\ \tilde{E}_i^{(1)} = \tilde{E}_i(t_n)\ , \\ \pi_a^{(1)} = \pi_a(t_n)\ , \end{cases}$$

$$\begin{cases} K_\phi^{(1)} = K_\phi(a^{(1)}, \tilde{\pi}_\phi^{(1)}) \\ V^{(1)} = V(\phi^{(1)}) \\ K_A^{(1)} = K_A(a^{(1)}, \tilde{E}_i^{(1)}) \\ G_A^{(1)} = G_A(a^{(1)}, A_i^{(1)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(1)} = \mathcal{K}_\phi\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_A^{(1)} = \mathcal{K}_A\left[\phi^{(1)}, \tilde{\pi}_\phi^{(1)}, a^{(1)}, \vec{\tilde{E}}^{(1)}, \vec{B}^{(1)}\right]\ , \\ \mathcal{K}_a^{(1)} = \mathcal{K}_a\left[a^{(1)}, K_\phi^{(1)}, V^{(1)}, K_A^{(1)}, G_A^{(1)}\right]\ . \end{cases}$$

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right]\ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \ . \end{cases}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$t_{n+1} \equiv (2)$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t\ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t\ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t\ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t\ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t\ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t\ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right]\ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right]\ . \end{cases}$$

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$
$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$
$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$
$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \;, \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \;, \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \;, \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \;, \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \;, \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \;. \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \;, \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \tilde{\vec{E}}^{(2)}, \vec{B}^{(2)}\right] \;, \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \;. \end{cases}$$

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$t_{n+1} \equiv (2)$$

$$\begin{cases} \phi^{(2)} = \phi^{(1)} + (a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)}\delta t \ , \\ A_i^{(2)} = A_i^{(1)} + (a^{(1)})^{-1}\tilde{E}_i^{(1)}\delta t \ , \\ a^{(2)} = a^{(1)} + \pi_a^{(1)}\delta t \ , \\ \tilde{\pi}_\phi^{(2)} = \tilde{\pi}_\phi^{(1)} + \mathcal{K}_\phi^{(1)}\delta t \ , \\ \tilde{E}_i^{(2)} = \tilde{E}_i^{(1)} + \mathcal{K}_A^{(1)}\delta t \ , \\ \pi_a^{(2)} = \pi_a^{(1)} + \mathcal{K}_a^{(1)}\delta t \ . \end{cases}$$

$$\begin{cases} K_\phi^{(2)} = K_\phi(a^{(2)}, \tilde{\pi}_\phi^{(2)}) \\ V^{(2)} = V(\phi^{(2)}) \\ K_A^{(2)} = K_A(a^{(2)}, \tilde{E}_i^{(2)}) \\ G_A^{(2)} = G_A(a^{(2)}, A_i^{(2)}) \end{cases}$$

$$\begin{cases} \mathcal{K}_\phi^{(2)} = \mathcal{K}_\phi\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_A^{(2)} = \mathcal{K}_A\left[\phi^{(2)}, \tilde{\pi}_\phi^{(2)}, a^{(2)}, \vec{\tilde{E}}^{(2)}, \vec{B}^{(2)}\right] \ , \\ \mathcal{K}_a^{(2)} = \mathcal{K}_a\left[a^{(2)}, K_\phi^{(2)}, V^{(2)}, K_A^{(2)}, G_A^{(2)}\right] \ . \end{cases}$$

$$\begin{aligned} \phi(t_{n+1}) &= \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2 \\ A_i(t_{n+1}) &= A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2 \\ a(t_{n+1}) &= a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2 \end{aligned}$$

$$\begin{aligned} \tilde{\pi}_\phi(t_{n+1}) &= \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t \\ \tilde{E}_i(t_{n+1}) &= \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t \\ \pi_a(t_{n+1}) &= \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t \end{aligned}$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$
$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$
$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$
$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

More intermediate kernels

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

$$\tilde{\pi}_\phi(t_{n+1}) = \tilde{\pi}_\phi(t_n) + \frac{1}{2}(\mathcal{K}_\phi^{(1)} + \mathcal{K}_\phi^{(2)})\delta t$$

$$\tilde{E}_i(t_{n+1}) = \tilde{E}_i(t_n) + \frac{1}{2}(\mathcal{K}_A^{(1)} + \mathcal{K}_A^{(2)})\delta t$$

$$\pi_a(t_{n+1}) = \pi_a(t_n) + \frac{1}{2}(\mathcal{K}_a^{(1)} + \mathcal{K}_a^{(2)})\delta t$$

Aux fields

More intermediate kernels

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$

$$A_i(t_{n+1}) = A_i(t_n) + \left[(a^{(1)})^{-1}\tilde{E}_i^{(1)} + (a^{(2)})^{-1}\tilde{E}_i^{(2)}\right]\delta t/2$$

$$a(t_{n+1}) = a(t_n) + (\pi_a^{(1)} + \pi_a^{(2)})\delta t/2$$

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More intermediate kernels $\rightarrow$ Aux fields $\rightarrow$ Memory disadvantages

Explicit-in-time integrator

Axion-Inflation using RK2

$$\phi(t_{n+1}) = \phi(t_n) + \left[(a^{(1)})^{-3}\tilde{\pi}_\phi^{(1)} + (a^{(2)})^{-3}\tilde{\pi}_\phi^{(2)}\right]\delta t/2$$
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More intermediate kernels → Aux fields → Memory disadvantages

More intermediate kernels → More steps

Explicit-in-time integrator

Axion-Inflation using RK2

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More intermediate kernels

Aux fields → Memory disadvantages

More steps → Time disadvantages

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Axion-Inflation using RK2

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More intermediate kernels → Aux fields → Memory disadvantages

More intermediate kernels → More steps → Time disadvantages

→ We use low storage RK methods

[D. G. Figueroa, A. Florio, T. Opferkuch, B. Stefanek (2112.08388)]

Axion-Inflation using RK2

Explicit-in-time integrator

Axion-Inflation using RK2

Dynamical equations

$$\mathcal{K}_\phi^L = a\sum_i \Delta_i^-\Delta_i^+\phi - a^3m^2\phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)}B_i^{(4)}$$

$$\mathcal{K}_A^L = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i}B_{i,+i}^{(4)}\right)$$

$$+\frac{1}{4a\Lambda}\sum_{\pm}\sum_{j,k}\epsilon_{ijk}\left\{\left[(\Delta_j^\pm\phi)\tilde{E}_{k,\pm j}^{(2)}\right]_{+i} + \left[(\Delta_j^\pm\phi)\tilde{E}_{k,\pm j}^{(2)}\right]\right\}$$

$$\mathcal{K}_a^L = -\frac{a}{6m_{pl}^2}(\rho_L + 3p_L)$$

Axion-Inflation using RK2

Dynamical equations

$$\mathcal{K}^L_\phi = a\sum_i \Delta_i^- \Delta_i^+ \phi - a^3 m^2 \phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)} B_i^{(4)}$$

$$\mathcal{K}^L_A = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B_{i,+i}^{(4)}\right)$$

$$+\frac{1}{4a\Lambda}\sum_\pm \sum_{j,k}\epsilon_{ijk}\left\{\left[(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}\right]_{+i} + \left[(\Delta_j^\pm \phi)\tilde{E}^{(2)}_{k,\pm j}\right]\right\}$$

$$\mathcal{K}^L_a = -\frac{a}{6m_{pl}^2}(\rho_L + 3p_L)$$

Constraints

$$\sum_i \Delta_i^- \tilde{E}_i = -\frac{1}{2\Lambda}\sum_\pm \sum_i \left(\Delta_i^\pm \phi\right) B_{i,\pm i}^{(4)}$$

$$\pi_a^2 = \frac{a^2}{3m_{pl}^2}\rho_L$$

Explicit-in-time integrator

Axion-Inflation using RK2

No temporal semi-sums

Dynamical equations

$$\mathcal{K}_\phi^L = a\sum_i \Delta_i^- \Delta_i^+ \phi - a^3 m^2 \phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)} B_i^{(4)}$$

$$\mathcal{K}_A^L = -\frac{1}{a}\sum_{j,k}\epsilon_{ijk}\Delta_j^- B_k - \frac{1}{2a^3\Lambda}\left(\tilde{\pi}_\phi B_i^{(4)} + \tilde{\pi}_{\phi,+i} B_{i,+i}^{(4)}\right)$$

$$+\frac{1}{4a\Lambda}\sum_\pm \sum_{j,k}\epsilon_{ijk}\left\{\left[(\Delta_j^\pm \phi)\tilde{E}_{k,\pm j}^{(2)}\right]_{+i} + \left[(\Delta_j^\pm \phi)\tilde{E}_{k,\pm j}^{(2)}\right]\right\}$$

$$\mathcal{K}_a^L = -\frac{a}{6m_{pl}^2}(\rho_L + 3p_L)$$

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Explicit-in-time integrator

Axion-Inflation using RK2

Dynamical equations

$$\mathcal{K}_\phi^L = a\sum_i \Delta_i^-\Delta_i^+\phi - a^3m^2\phi + \frac{1}{2a\Lambda}\sum_i \tilde{E}_i^{(2)}B_i^{(4)}$$

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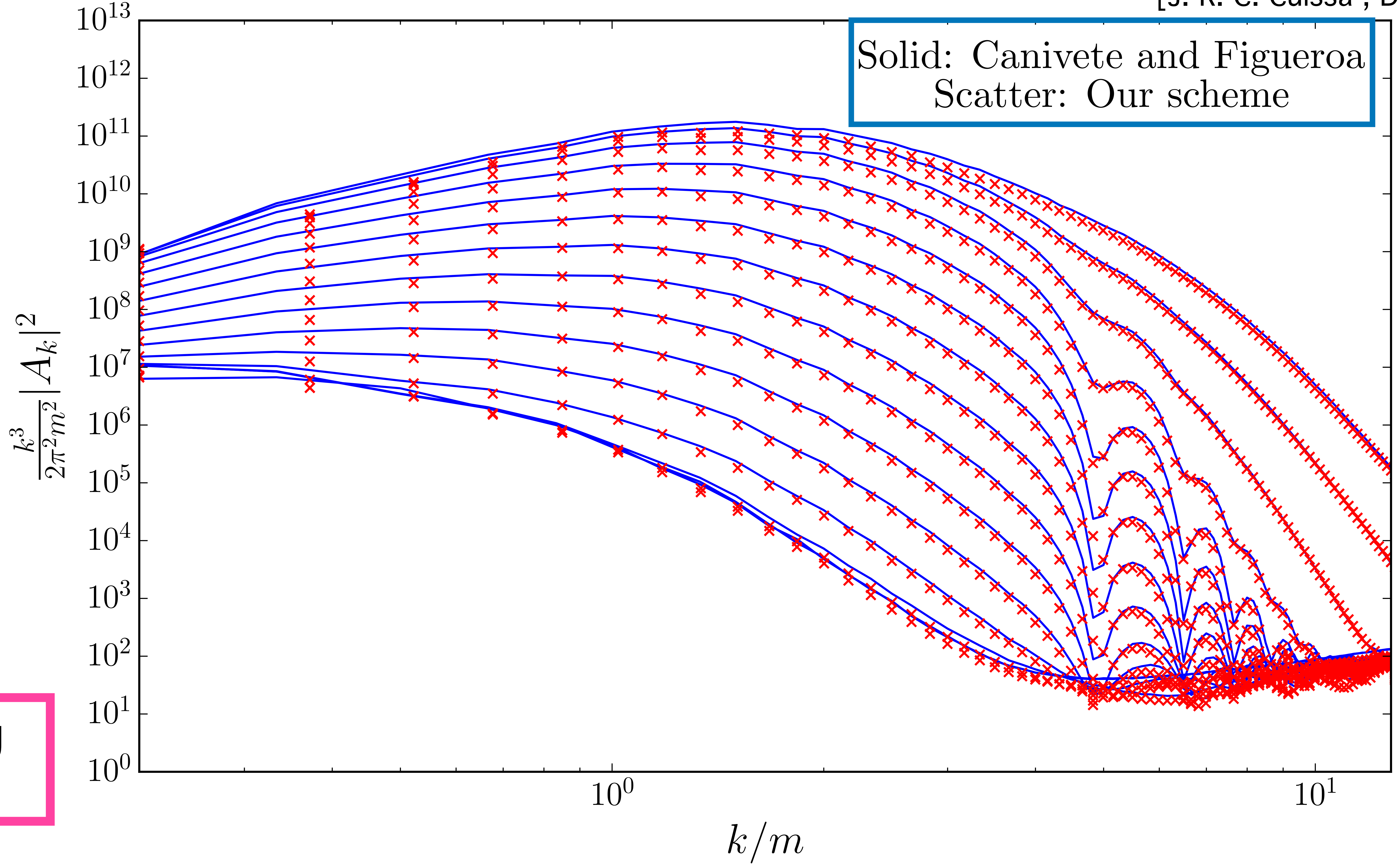
Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]

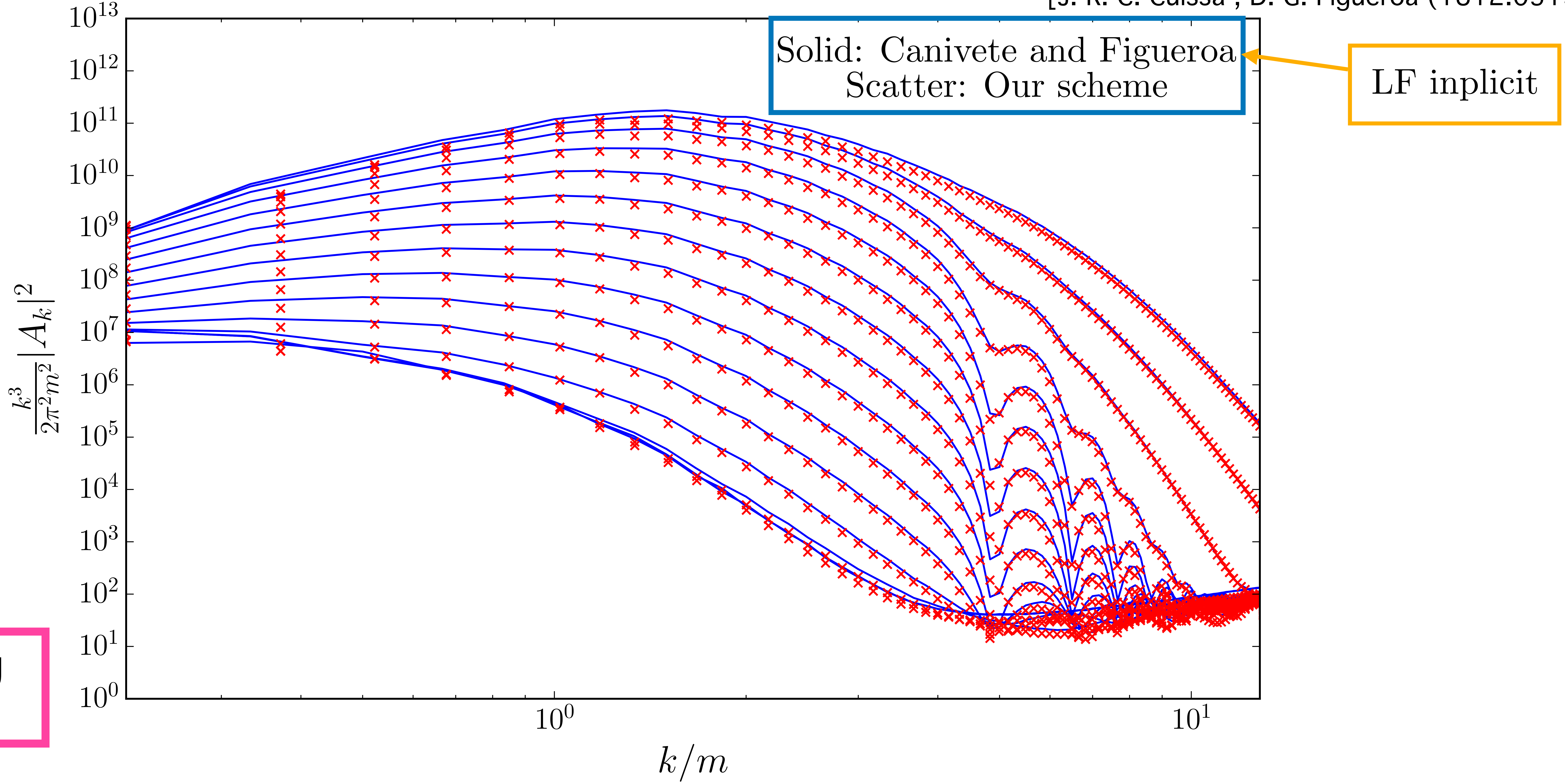


Preheating example

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]



Preheating example

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

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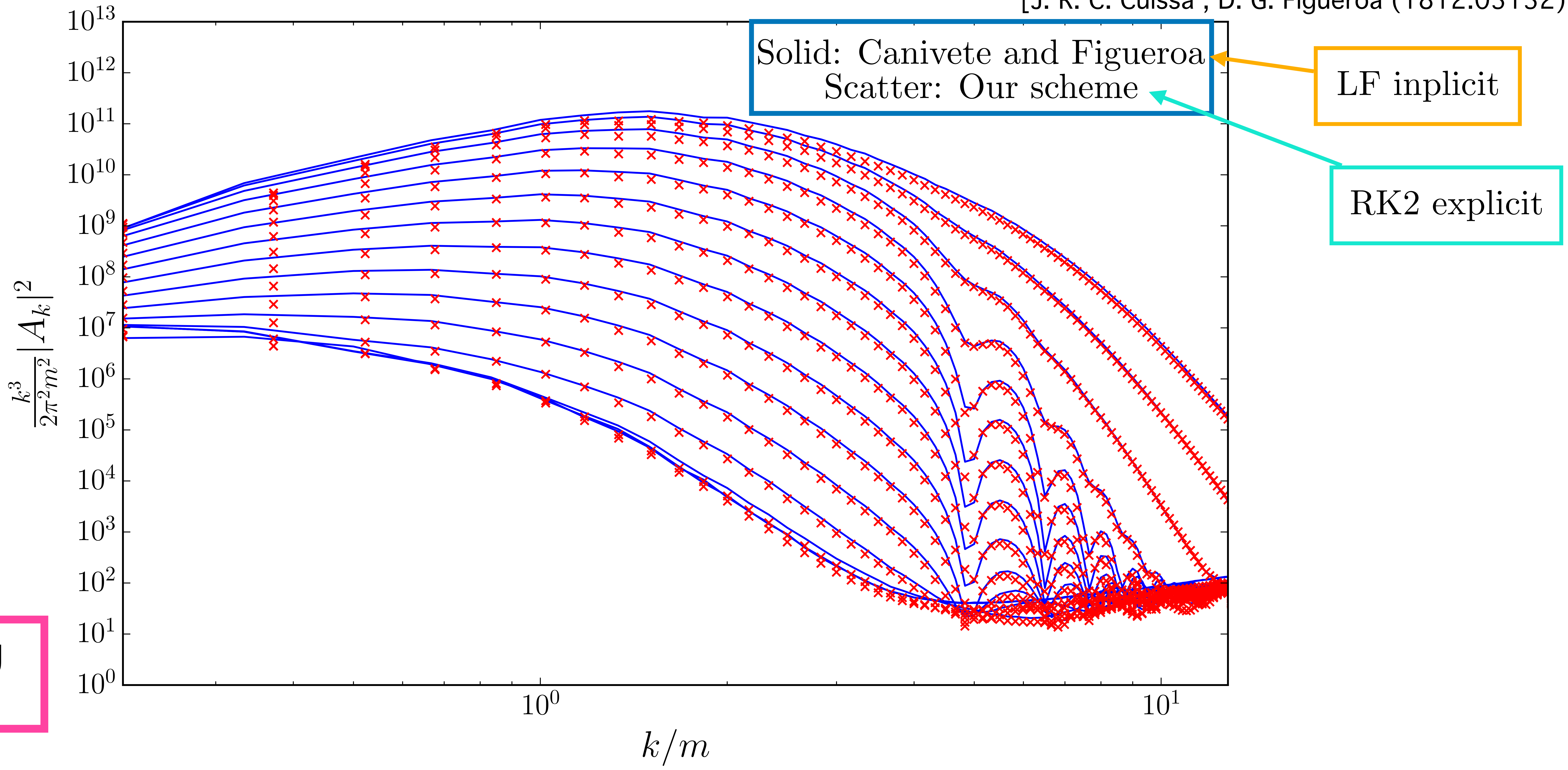

Preheating example

Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. G. Figueroa (1812.03132)]

Preheating example

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Preheating example

Explicit-in-time integrator

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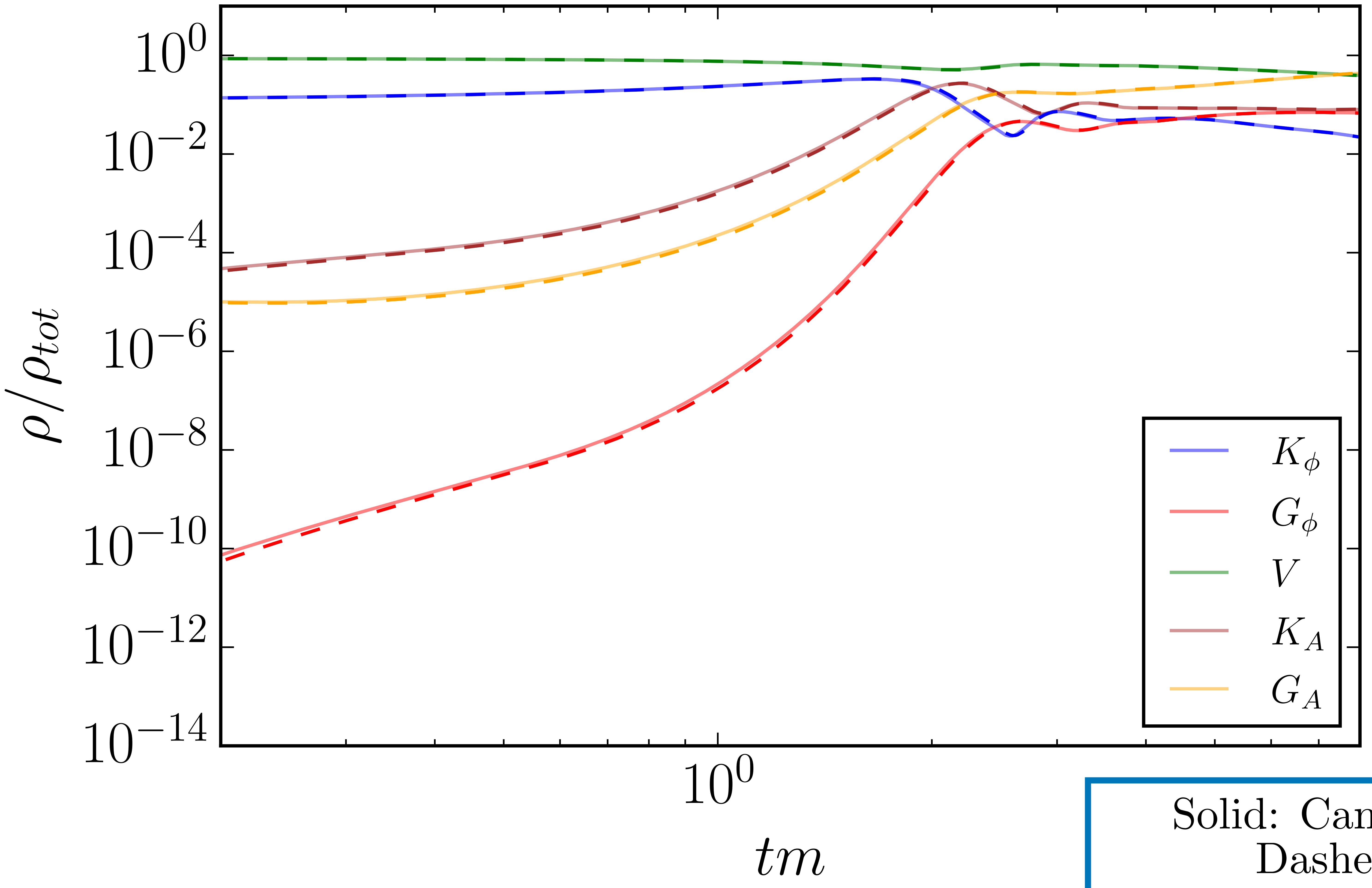


Preheating example

Solid: Canivete and Figueroa
Dashed: Our scheme

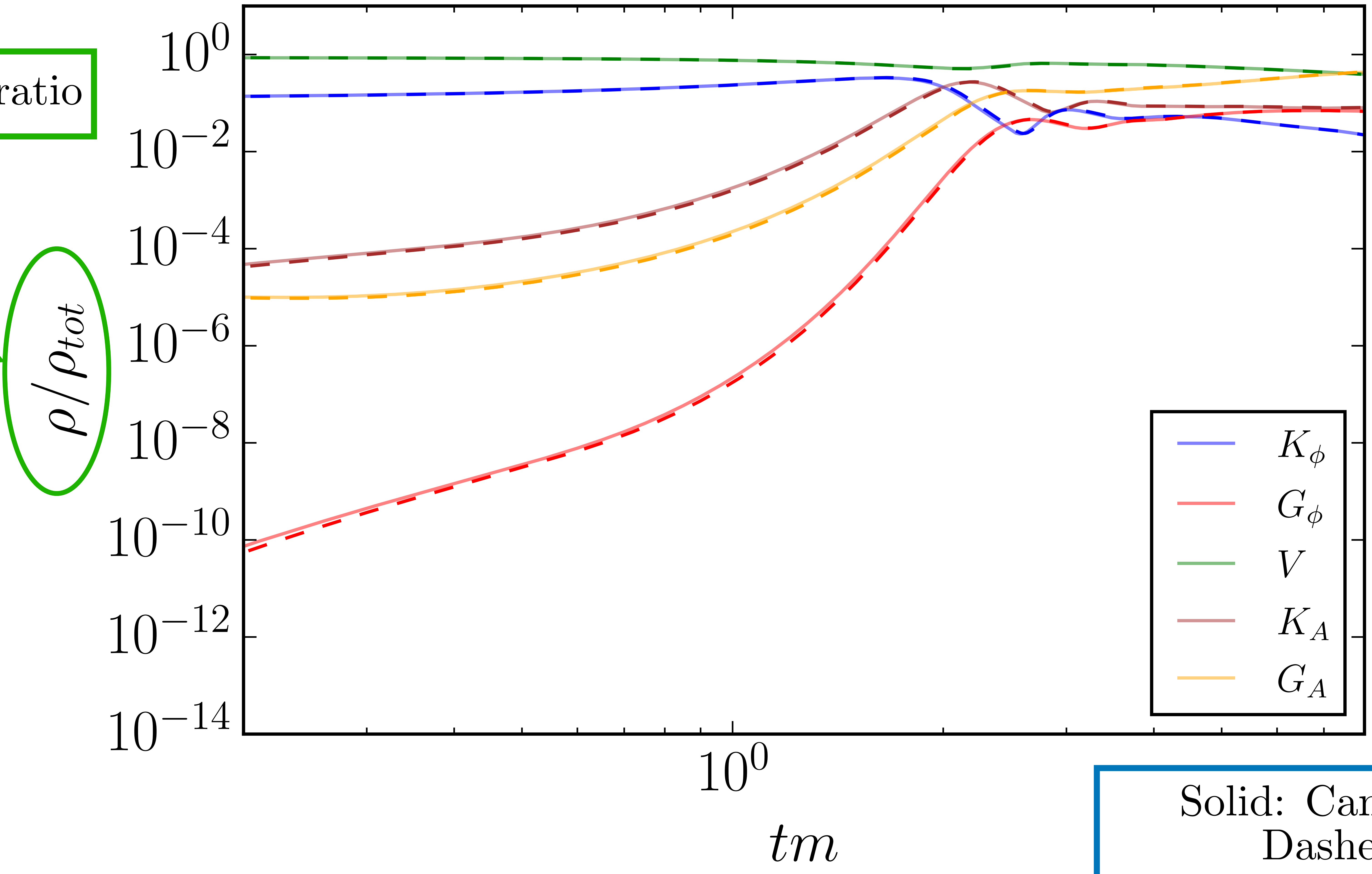
Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

Preheating example

Energy density ratio



Solid: Canivete and Figueroa
Dashed: Our scheme

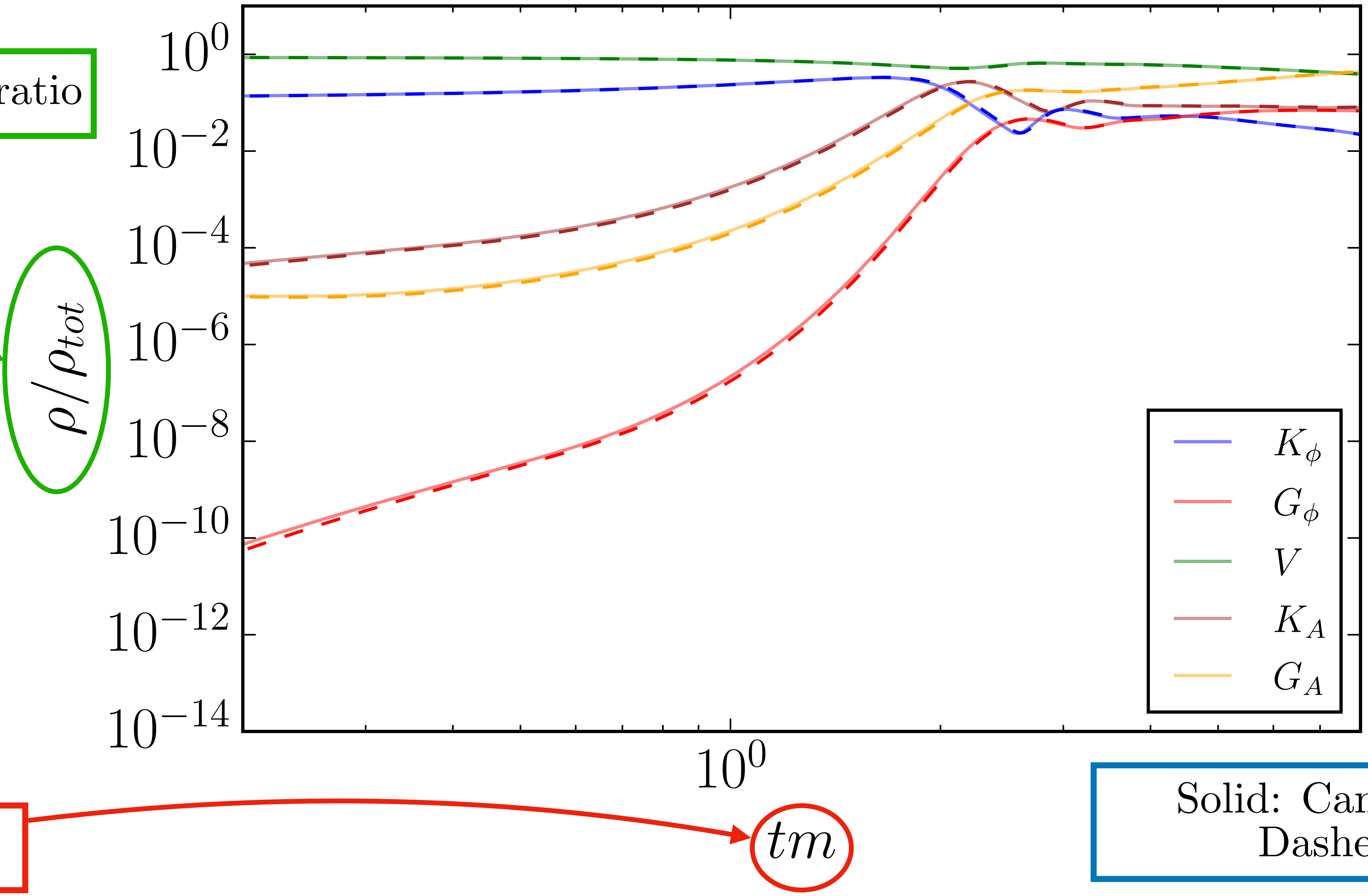
Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

[J. R. C. Cuissa , D. .G. Figueroa (1812.03132)]

Preheating example

Energy density ratio

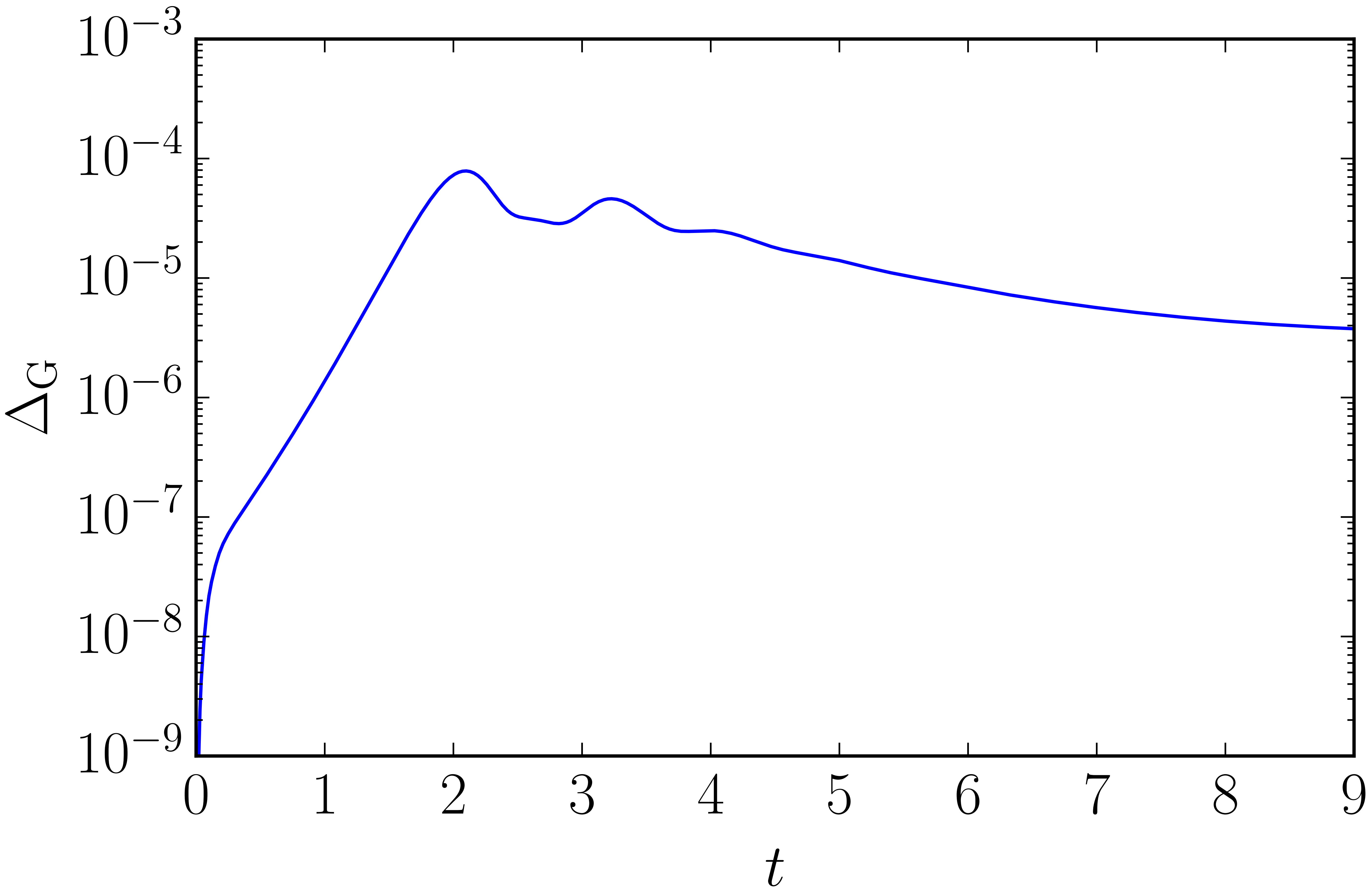


Cosmic time

Solid: Canivete and Figueroa
Dashed: Our scheme

Explicit-in-time integrator

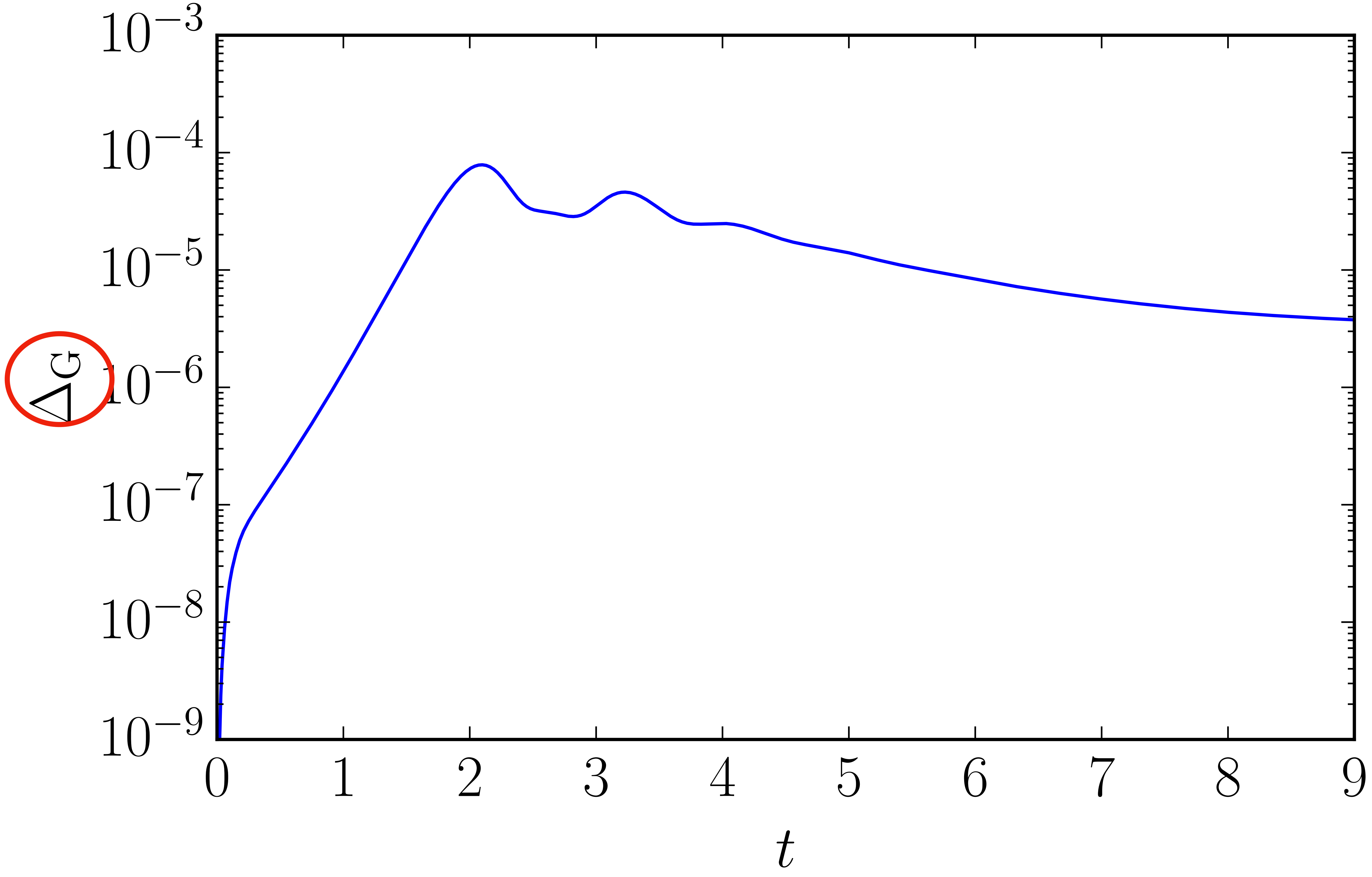
Axion-Inflation using RK2: validation of the scheme



Explicit-in-time integrator

Axion-Inflation using RK2: validation of the scheme

Gauss constraint observable



Initial Conditions in the Linear Regime

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla} \times \vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi \vec{B}$$

Initial Conditions in the Linear Regime

$$\dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla} \times \vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi \vec{B}$$

Helicity basis

Conformal time

Initial Conditions in the Linear Regime

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Helicity basis

Conformal time

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

Initial Conditions in the Linear Regime

$$\xi \equiv \frac{\dot{\phi}}{2H\Lambda}$$

$$\dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} \xrightarrow[\text{Conformal time}]{\text{Helicity basis}} \left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right)A_\pm(k,\tau) = 0$$

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$$|A^+| \gg |A^-|$$

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$$\left(\frac{\partial^2}{\partial \tau^2} + k^2 + \frac{2k\xi}{\tau}\right) A_+(k,\tau) = 0$$

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Deep inside Hubble $k \gg aH$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 + \frac{2k\xi}{\tau}\right) A_+(k,\tau) = 0$$

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Deep inside Hubble $k \gg aH$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2\right)A_+(k,\tau) \simeq 0$$

Initial Conditions in the Linear Regime

$$\dot{\vec{\tilde{E}}} = -\frac{1}{a}\vec{\nabla} \times \vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi \vec{B}$$

Helicity basis

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Bunch-Davies solution

Deep inside Hubble $k \gg aH$

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Initial Conditions in the Linear Regime

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$$A_+(k,\tau) \simeq \frac{1}{\sqrt{2k}}e^{-ik\tau}$$

Cosmic time

$\tau \simeq -1/aH$

Initial Conditions in the Linear Regime

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Bunch-Davies solution

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Cosmic time

$\tau \simeq -1/aH$

$$A_+(k,t) \simeq \frac{1}{\sqrt{2k}} e^{ik/a(t)H(t)}$$

Initial Conditions in the Linear Regime

$$\xi \equiv \frac{\dot{\phi}}{2H\Lambda}$$

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Cosmic time

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$$A_+(k,t) \simeq \frac{1}{\sqrt{2k}} e^{ik/a(t)H(t)}$$

Time derivative

Initial Conditions in the Linear Regime

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Helicity basis

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$$\left(\frac{\partial^2}{\partial\tau^2} + k^2\right)A_+(k,\tau) \simeq 0$$

Bunch-Davies solution

$$A_+(k,\tau) \simeq \frac{1}{\sqrt{2k}}e^{-ik\tau}$$

Cosmic time

$\tau \simeq -1/aH$

$$A_+(k,t) \simeq \frac{1}{\sqrt{2k}}e^{ik/a(t)H(t)}$$

Time derivative

$$E_+(k,t) \simeq \frac{i}{a}\sqrt{\frac{k}{2}}e^{ik/a(t)H(t)}$$

Initial Conditions in the Linear Regime

$$\xi \equiv \frac{\dot{\phi}}{2H\Lambda}$$

$$\dot{\vec{E}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi \vec{B}$$

Helicity basis

Conformal time

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

$|A^+| \gg |A^-|$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 + \frac{2k\xi}{\tau}\right) A_+(k,\tau) = 0$$

Deep inside Hubble $k \gg aH$

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2\right) A_+(k,\tau) \simeq 0$$

Bunch-Davies solution

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Cosmic time

$\tau \simeq -1/aH$

$$A_\pm(k,t) \simeq \frac{1}{\sqrt{2k}} e^{ik/a(t)H(t)}$$

Time derivative

$$E_\pm(k,t) \simeq \frac{i}{a}\sqrt{\frac{k}{2}}\, e^{ik/a(t)H(t)}$$

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}\,, E^{\pm}$

Initial Conditions in the Linear Regime

Helicity basis		Cartesian basis
BD solutions for $A^{\pm}\,,E^{\pm}$	$\longrightarrow$	CosmoLattice $A\,,E$

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$ $\longrightarrow$ CosmoLattice A , E

Cartesian basis

Objective

BD solution for each helicity

starting from cartesian basis

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$

Cartesian basis

CosmoLattice A , E

Objective

BD solution for each helicity

starting from cartesian basis

2 options

Initial Conditions in the Linear Regime

Helicity basis

BD solutions for $A^{\pm}$, $E^{\pm}$

Cartesian basis

CosmoLattice A , E

Objective

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Helicity vectors

$$k_i \varepsilon_i^{\pm}(\hat{k}) = 0 \; , \;\; \epsilon_{ijk} k_j \varepsilon^{\pm}(\hat{k}) = \mp i k \varepsilon_i^{\pm}(\hat{k}) \; ,$$

$$\varepsilon_i^{\pm}(\hat{k})^* = \varepsilon_i^{\pm}(-\hat{k}) \; , \;\; \varepsilon_i^{\lambda}(\hat{k}) \varepsilon_i^{\lambda'}(-\hat{k}) = \delta_{\lambda\lambda'}$$

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Projectors ✓

Use helicity projectors

$$A_i^{\pm} = P_{ij}^{\pm} A_j$$

Initial Conditions in the Linear Regime

How to build the Helicity Projector?

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2 eigenvalues: 1 and -1

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$$P^{\pm}_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2} \pm \frac{i}{k}\epsilon_{ijk}k_k\right) \longrightarrow \boxed{\begin{aligned} A^{\pm}_i &= P^{\pm}_{ij}A_j \\ E^{\pm}_i &= P^{\pm}_{ij}E_j \end{aligned}} \longrightarrow \vec{\nabla}\cdot\vec{E} = 0$$

Gauss

Two IC options

Initial Conditions in the Linear Regime

Two IC options

One option

Only project the $+$ helicity states

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Why?

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A^{+} is exponentially enhanced

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Our option

Project both states + and -

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Can A^{-} become relevant?

Initial Conditions in the Linear Regime

Gauge initialialization process

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$$A_i \to \frac{1}{\sqrt{2k}} e^{ik/aH}$$

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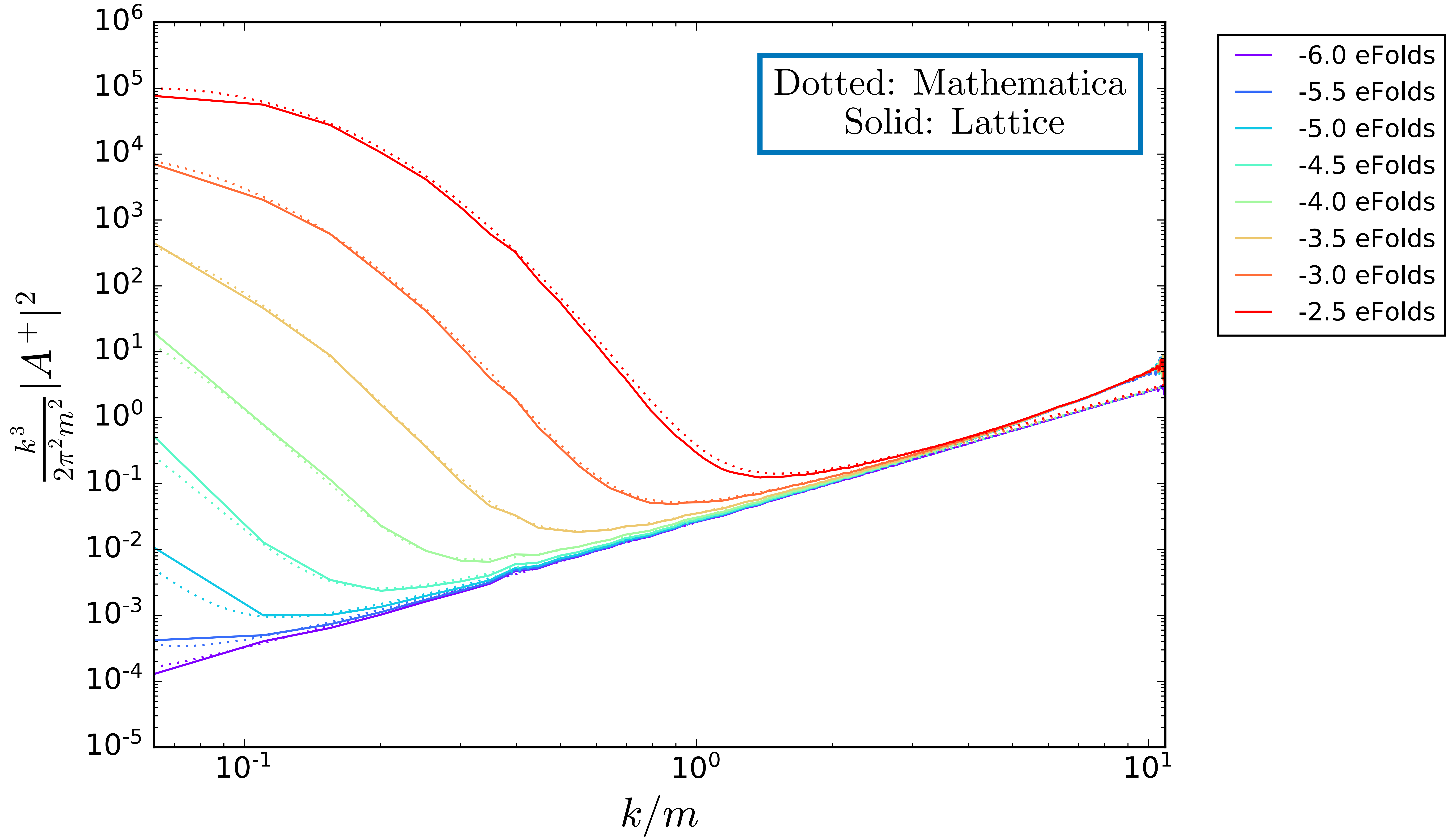
We will have an amplitude of 2 BD in A_i

$$\sum_i |A_i|^2 = |A_+|^2 + |A_-|^2$$

Simulations in the Linear Regime



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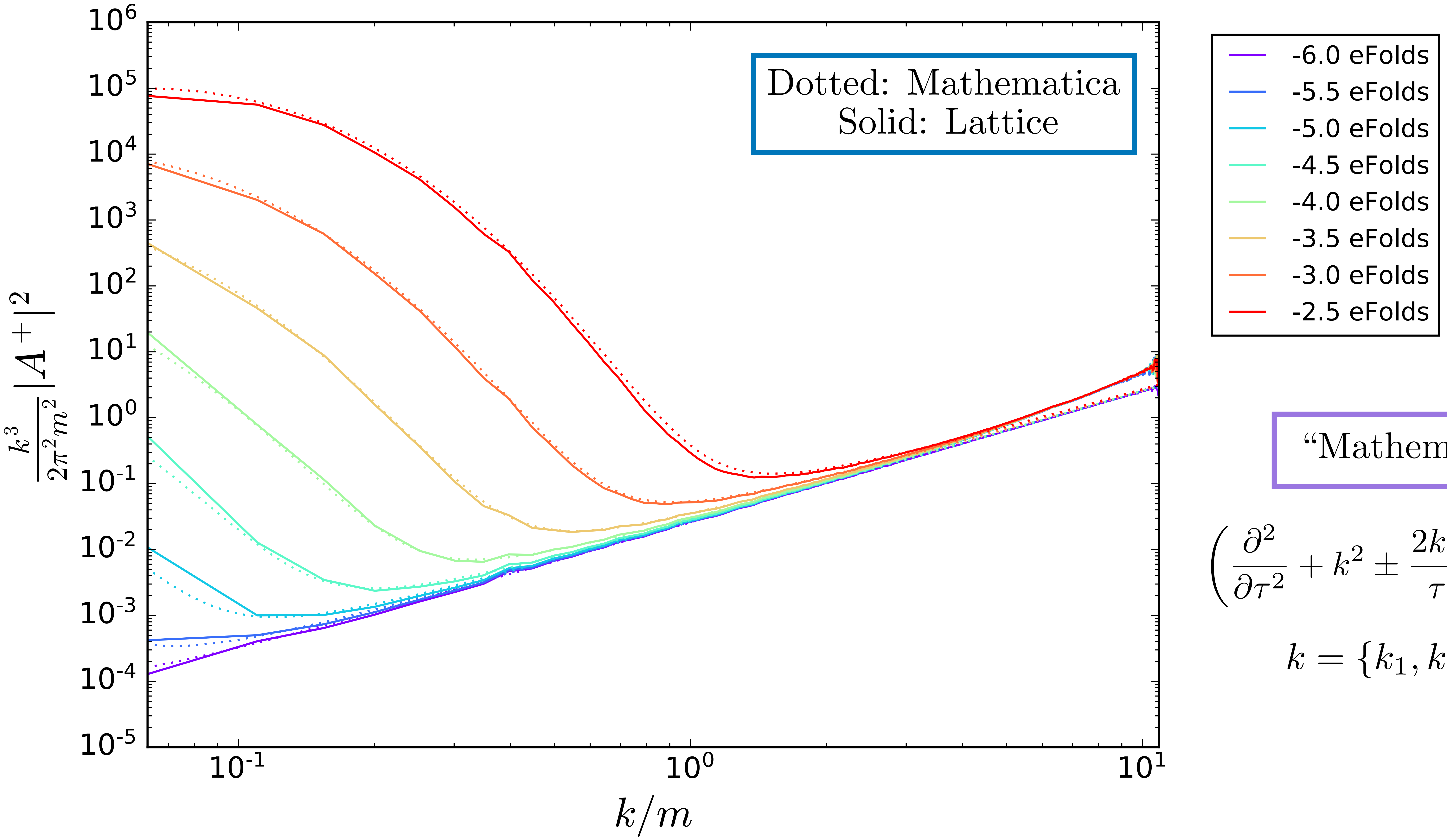
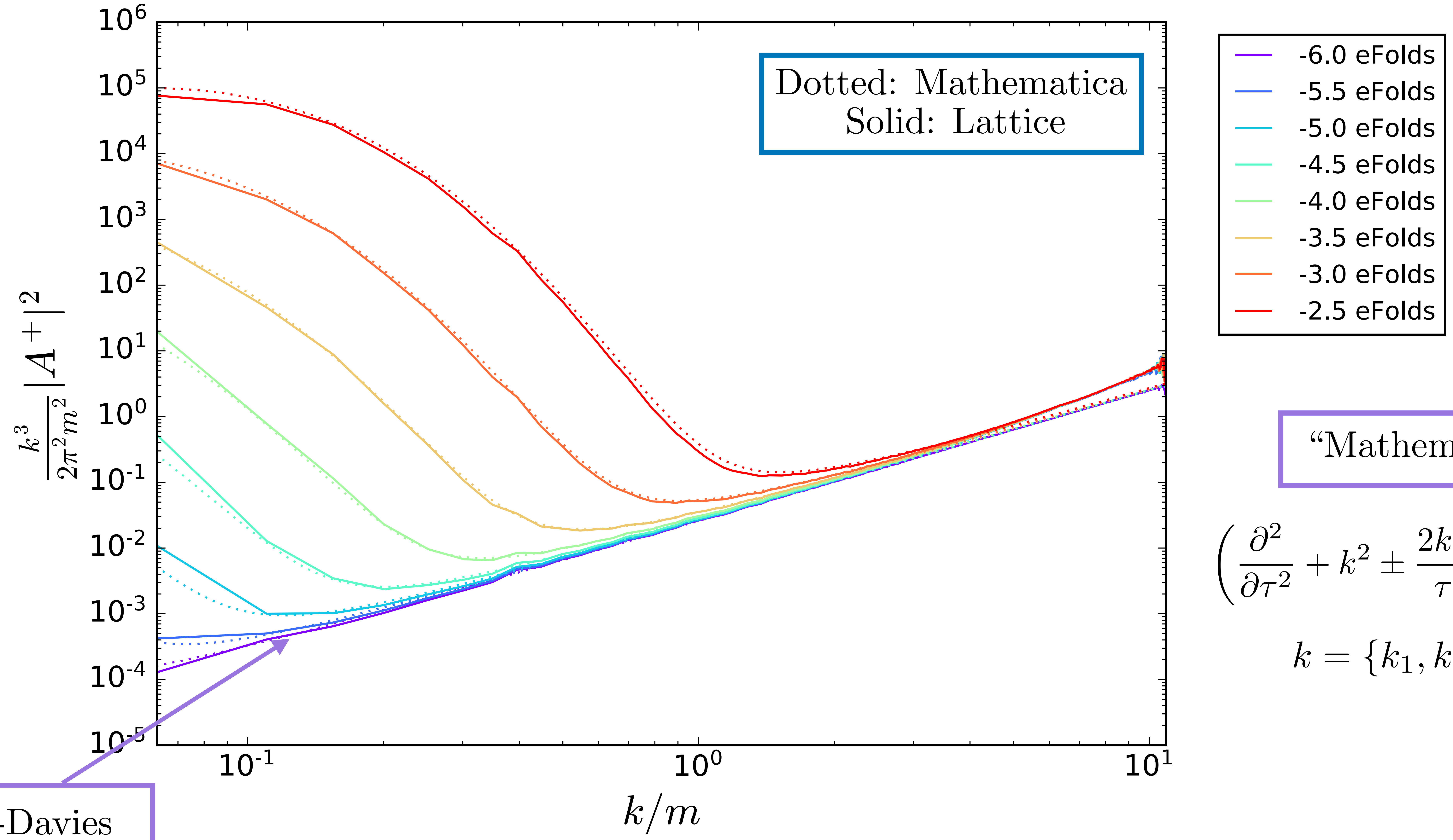


"Mathematica"

$$\left(\frac{\partial^2}{\partial\tau^2} + k^2 \pm \frac{2k\xi}{\tau}\right) A_\pm(k,\tau) = 0$$

$$k = \{k_1, k_2, ..., k_N\}$$

Simulations in the Linear Regime



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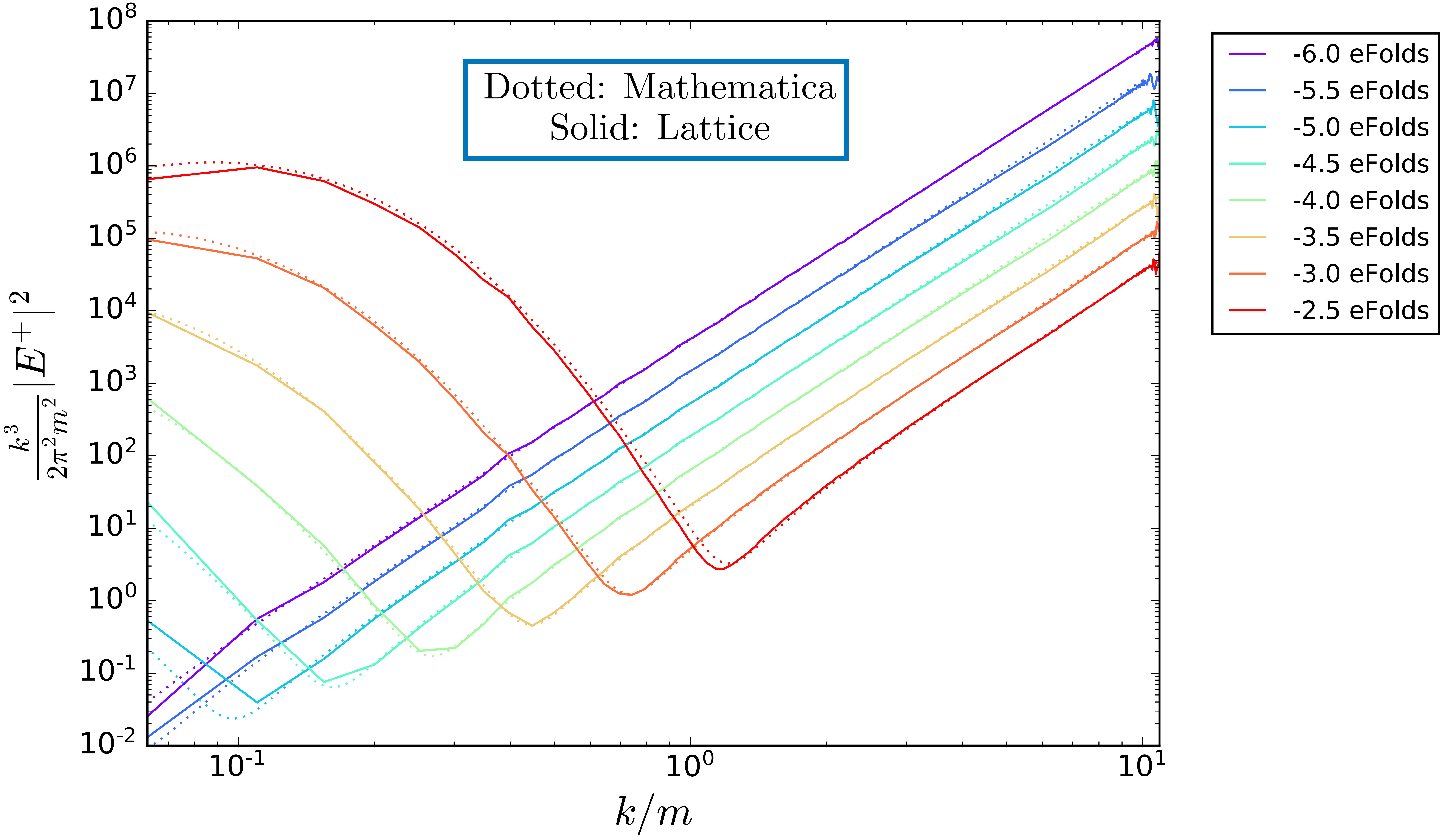

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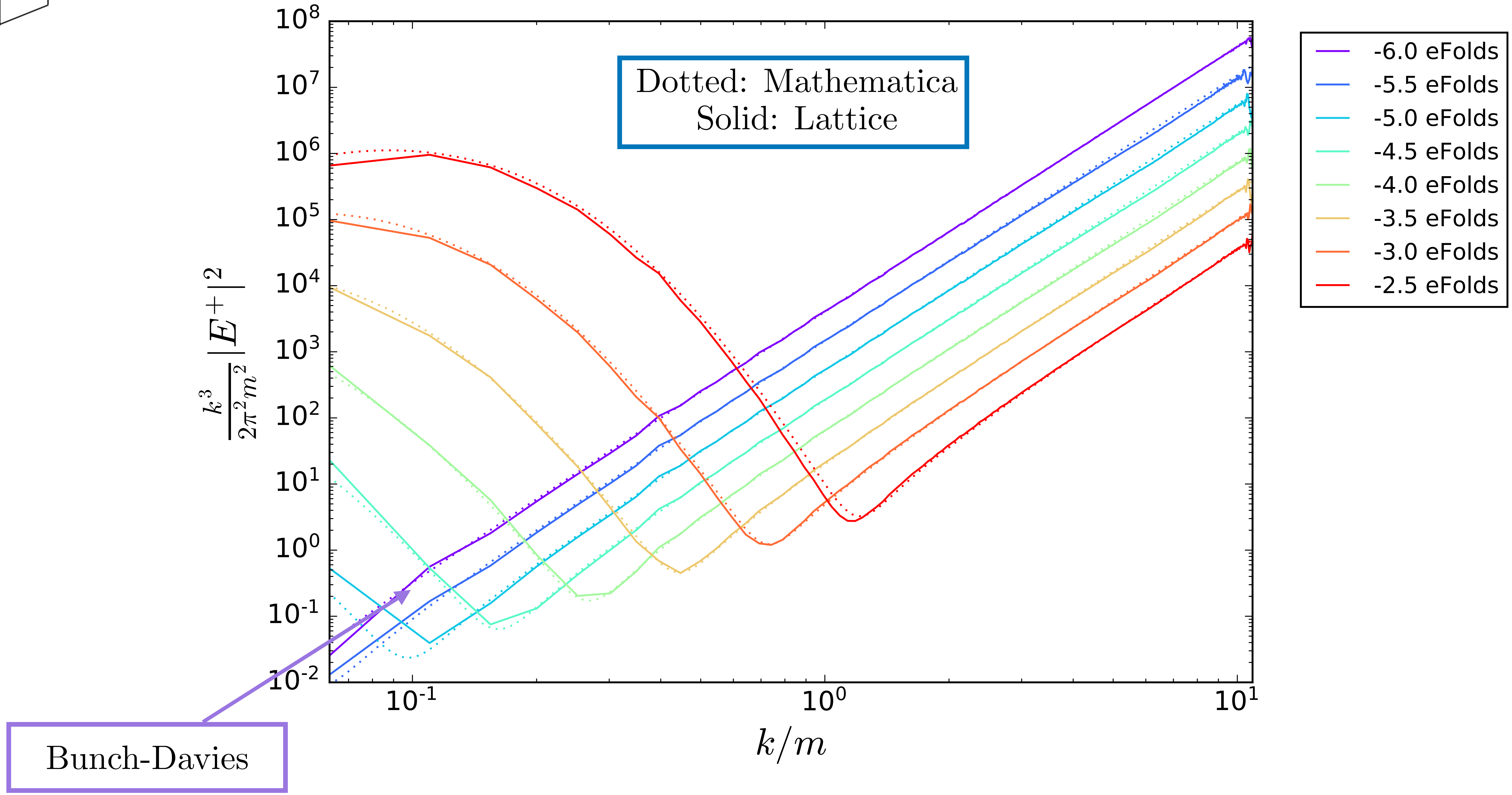

Non-linear evolution: example

Dynamical equations:

$$\dot{\tilde{\pi}}_\phi = a\vec{\nabla}^2\phi - a^3 m^2\phi + \frac{1}{a\Lambda}\tilde{\vec{E}}\cdot\vec{B}\ ,$$

$$\dot{\tilde{\vec{E}}} = -\frac{1}{a}\vec{\nabla}\times\vec{B} - \frac{1}{a^3\Lambda}\tilde{\pi}_\phi\vec{B} + \frac{1}{a\Lambda}\vec{\nabla}\phi\times\tilde{\vec{E}}$$

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Expansion equation:

$$\dot{\pi}_a = \frac{-a}{6m_{\rm pl}^2}(3p+\rho) =$$

$$\frac{a}{3m_{\rm pl}^2}\langle -2K_\phi + V - K_A - G_A \rangle$$

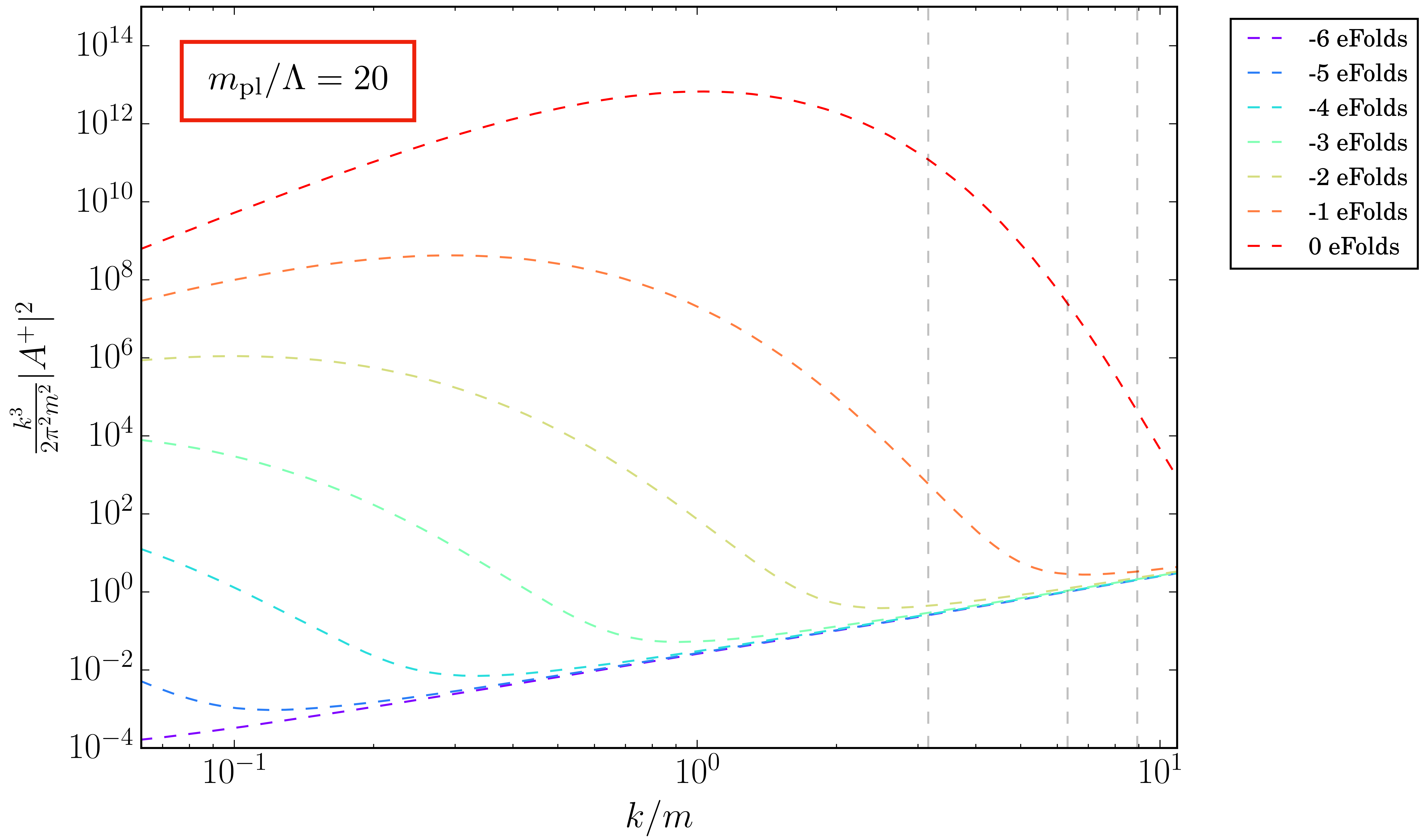
@

Non-linear evolution: example

Gauge PS

A^+

Linear



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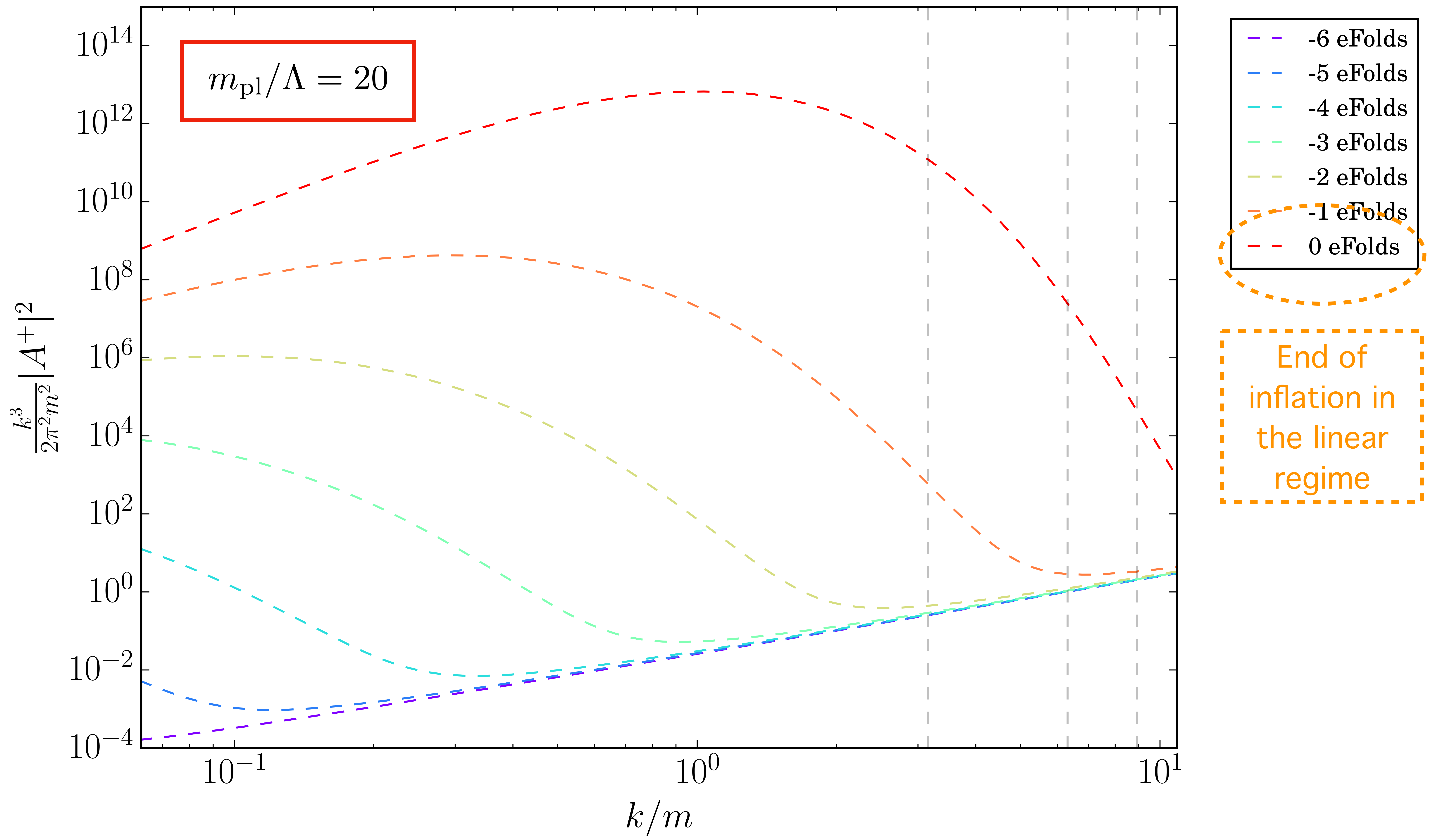

End of inflation in the linear regime

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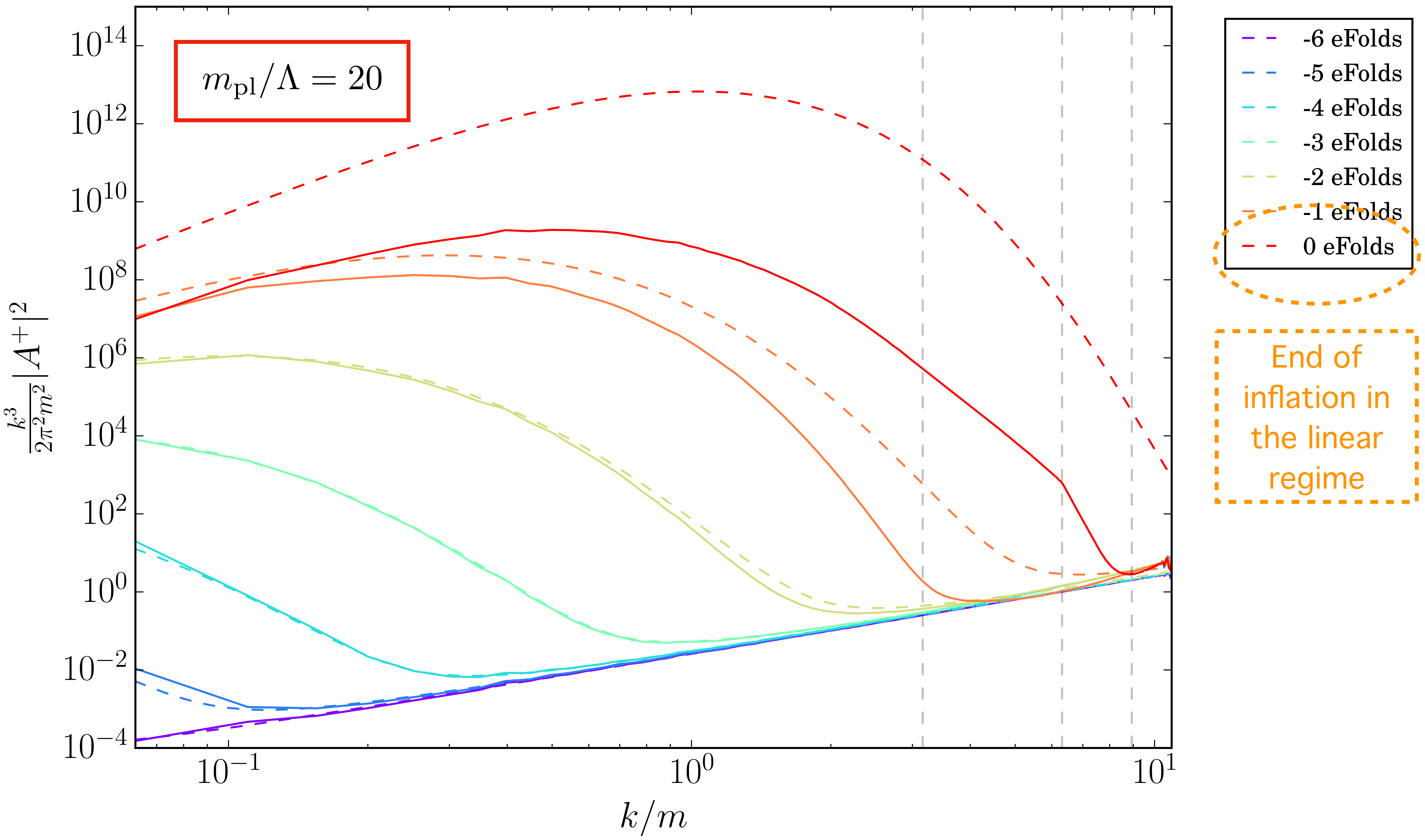

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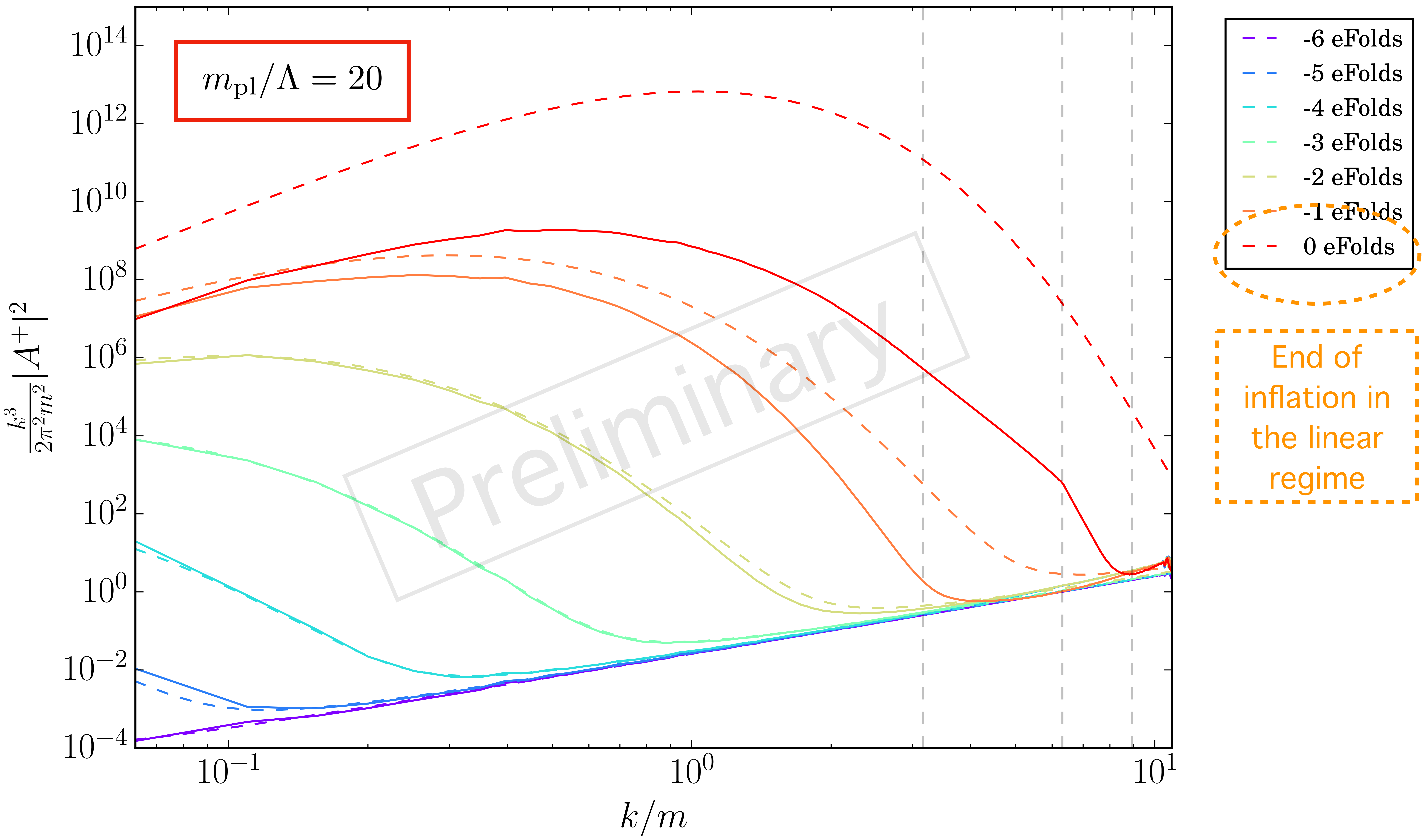

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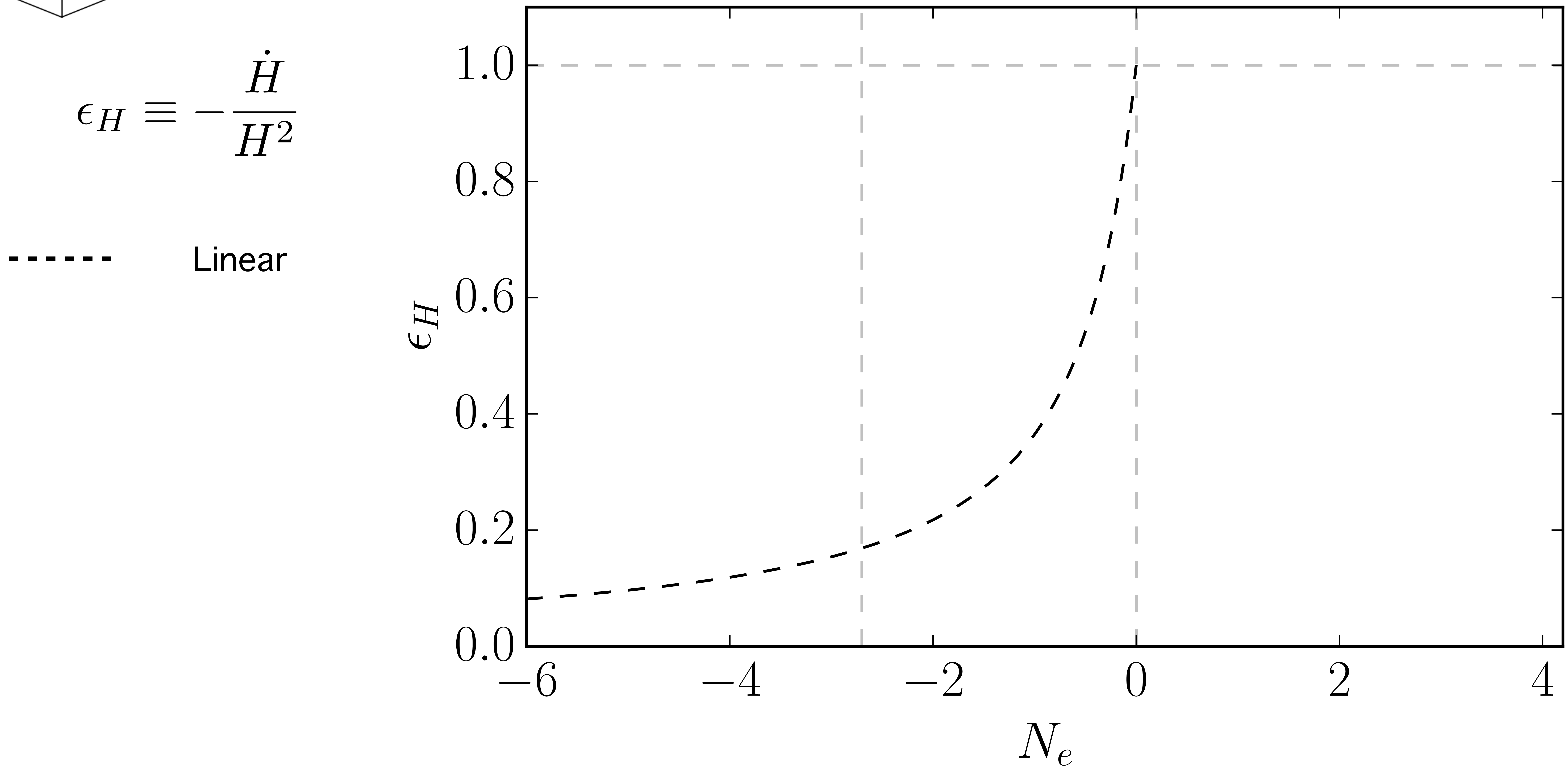

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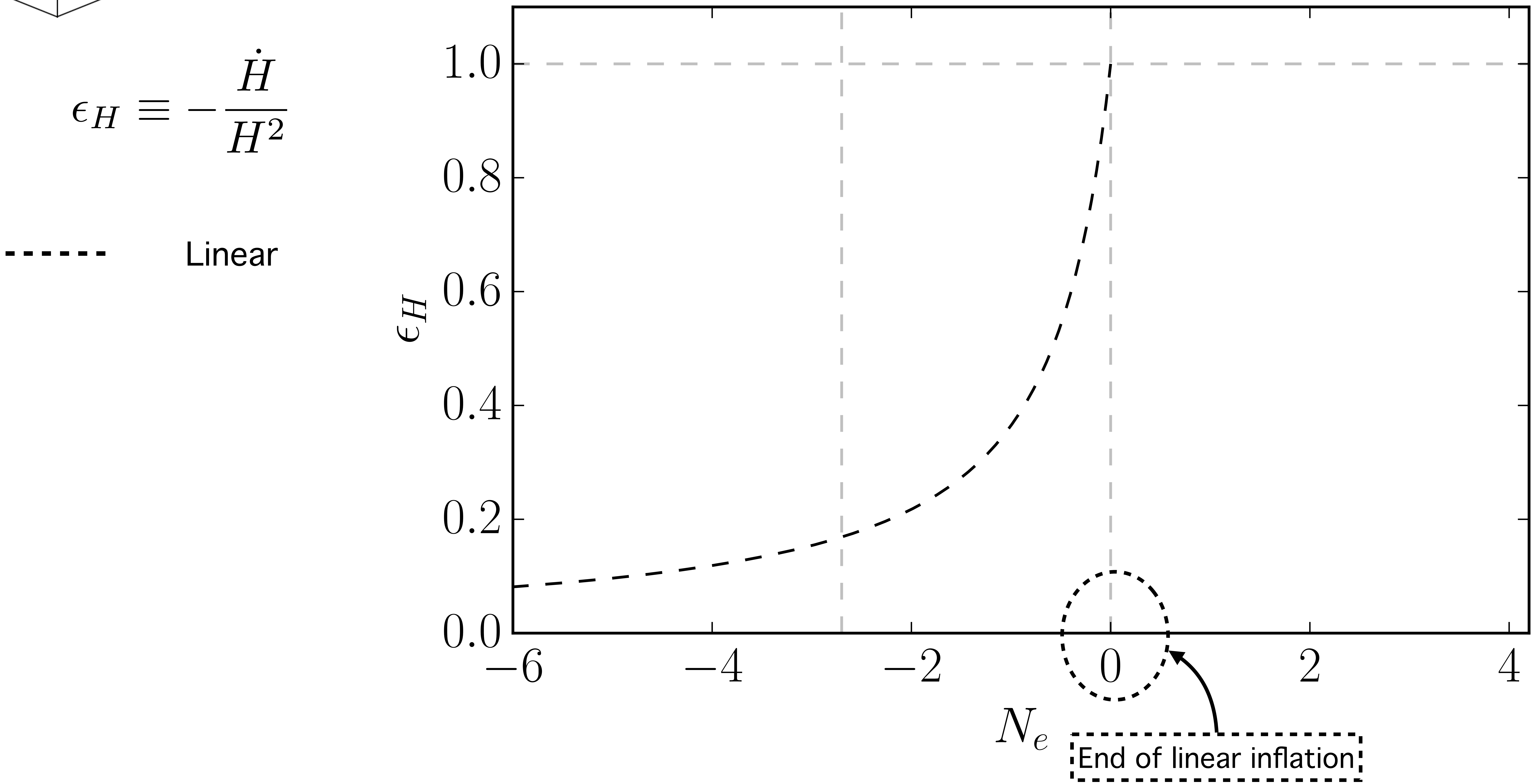

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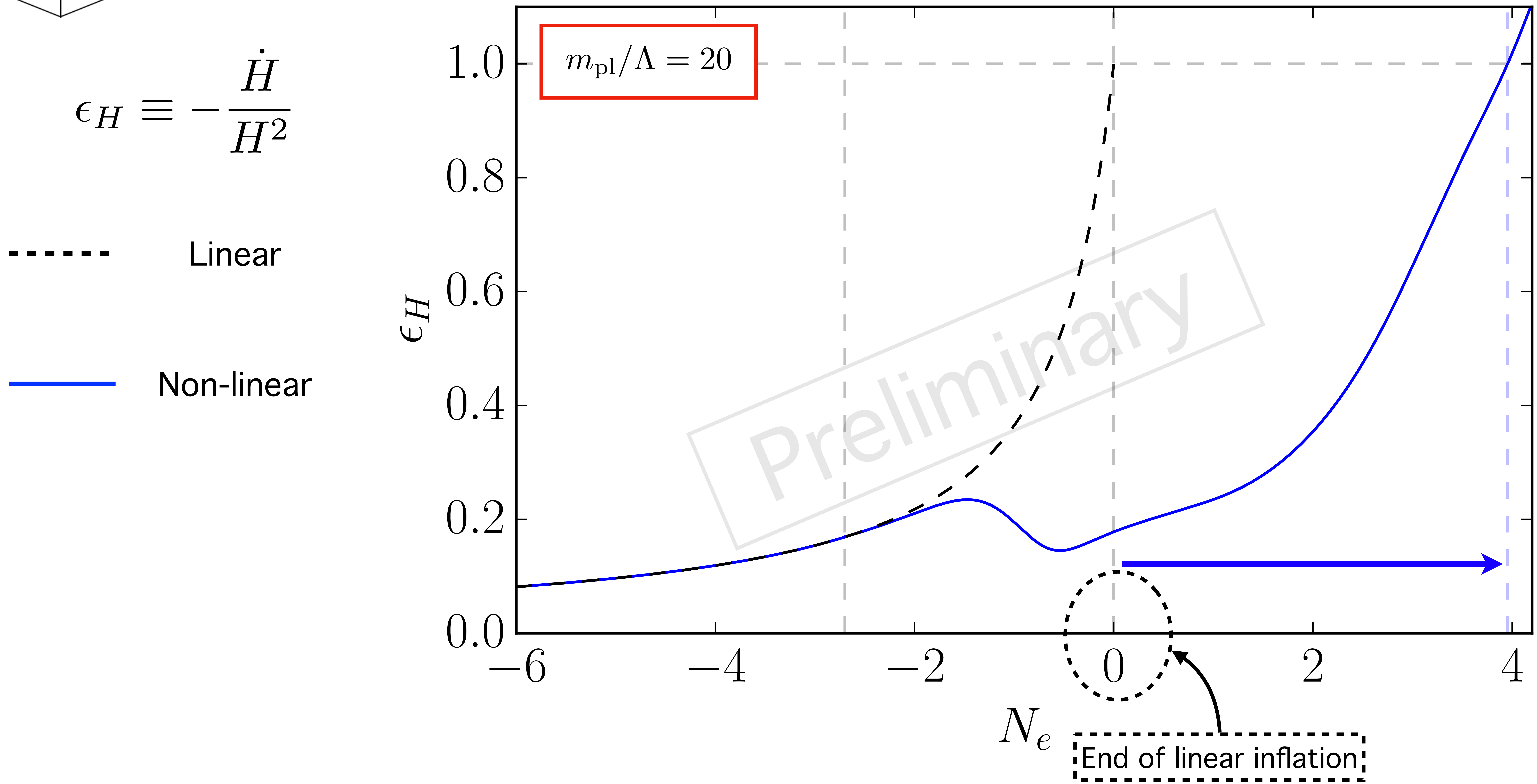

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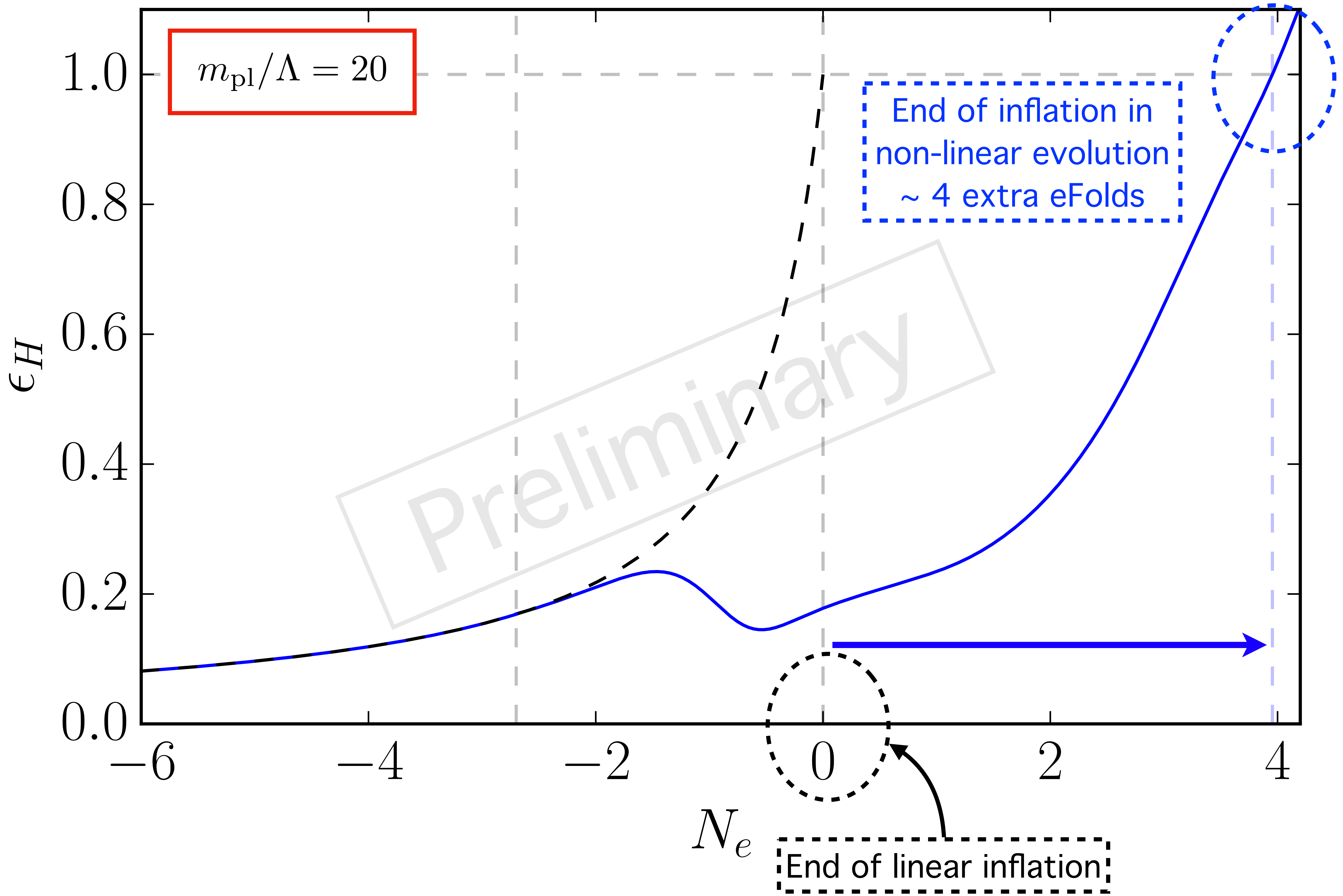
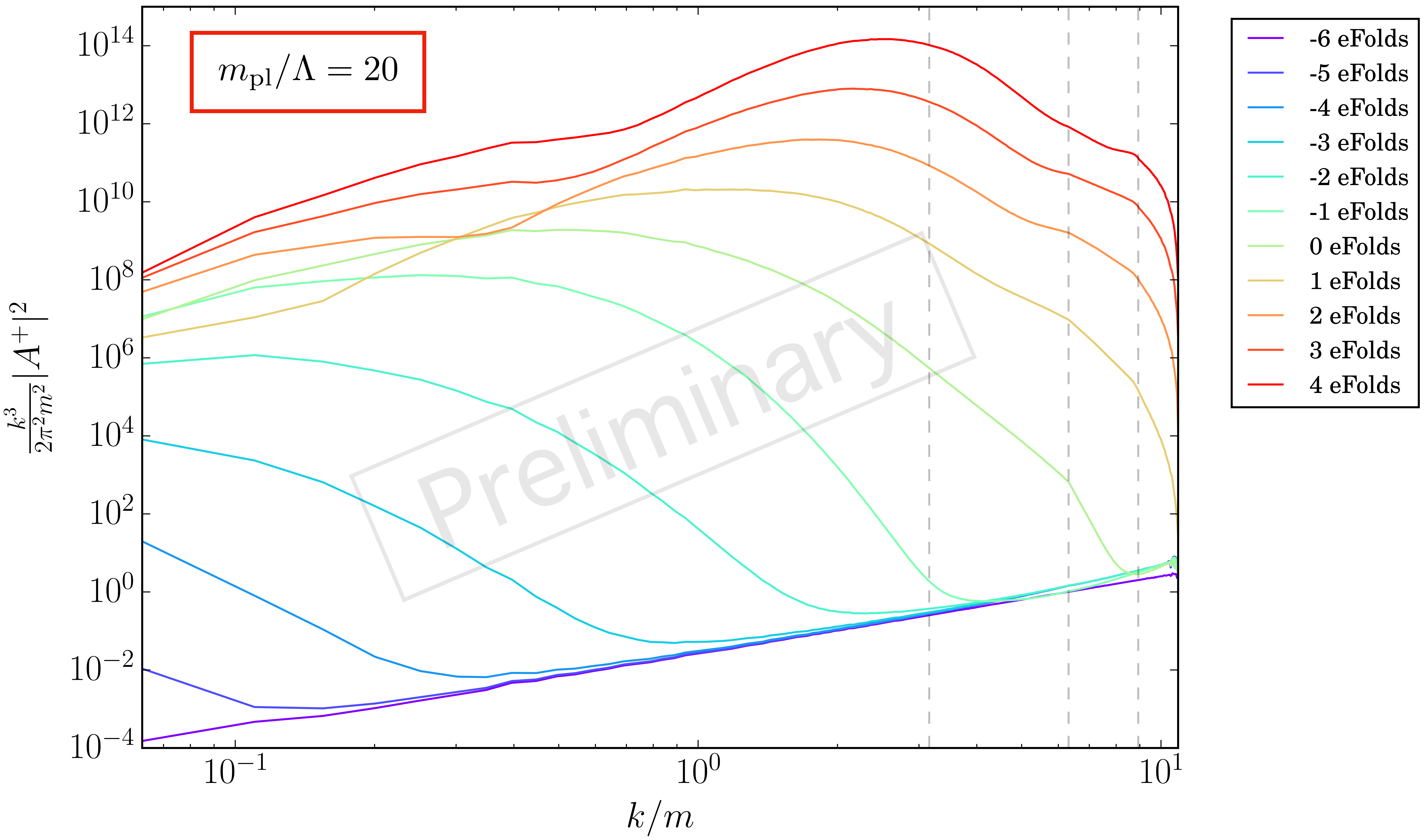


Non-linear evolution: example

What's happening in those extra folds?



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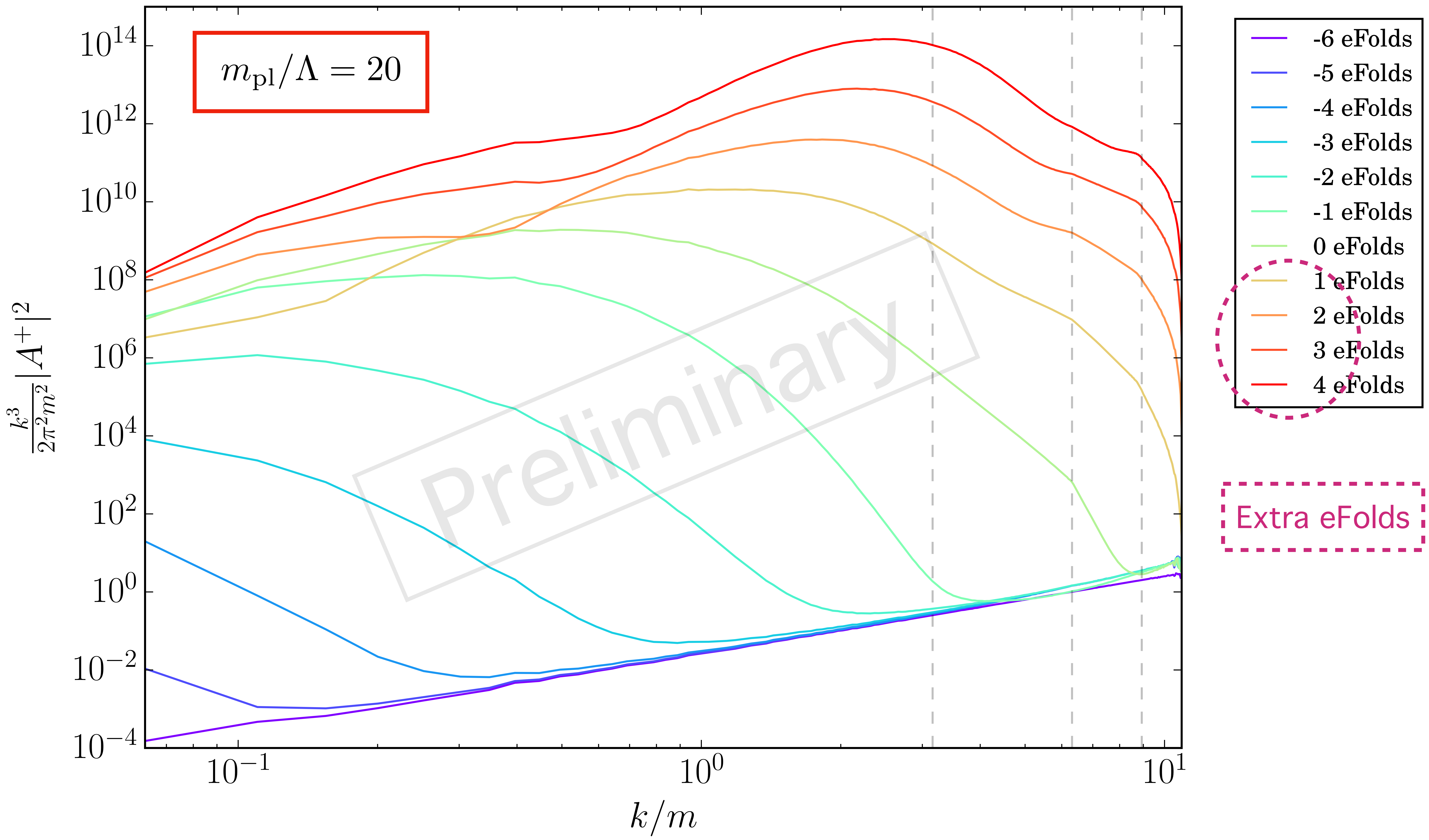

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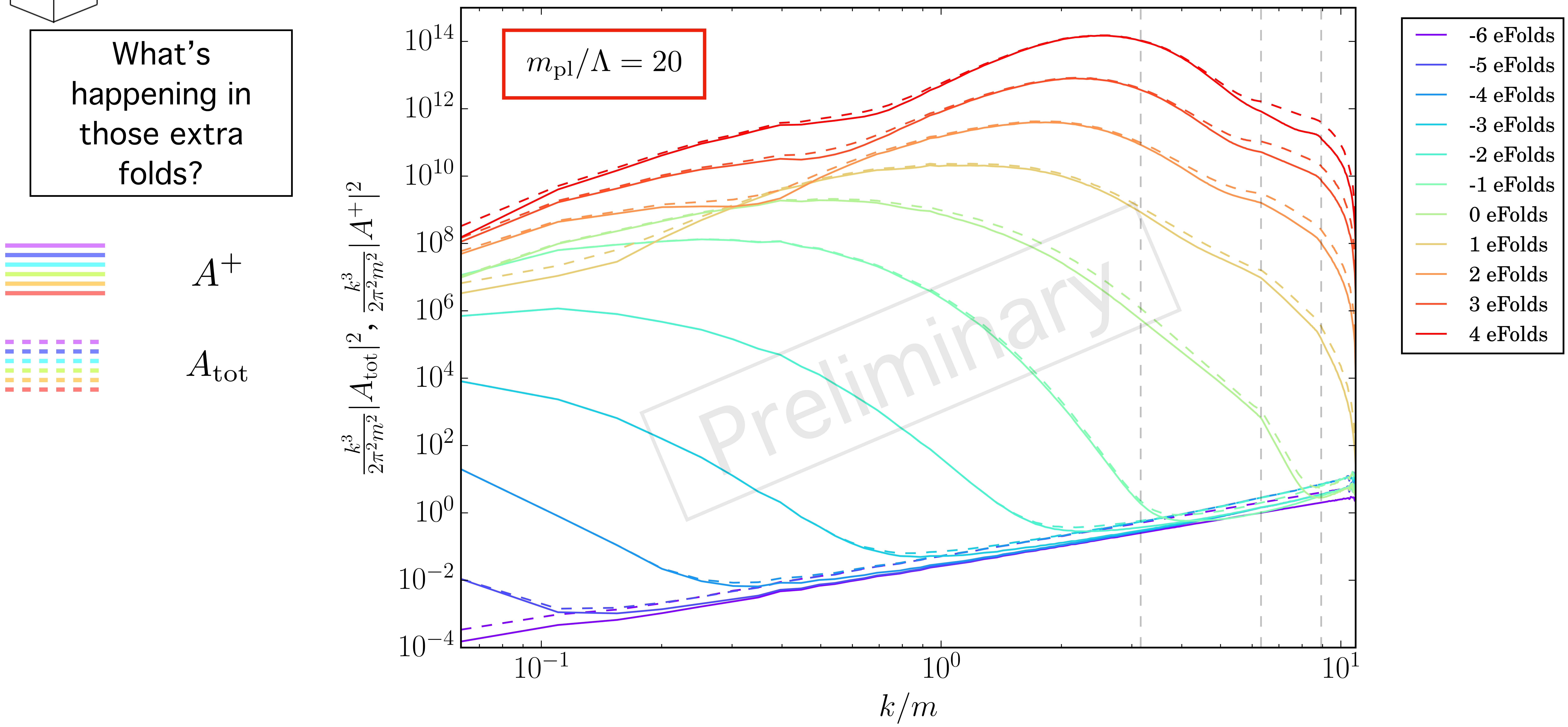

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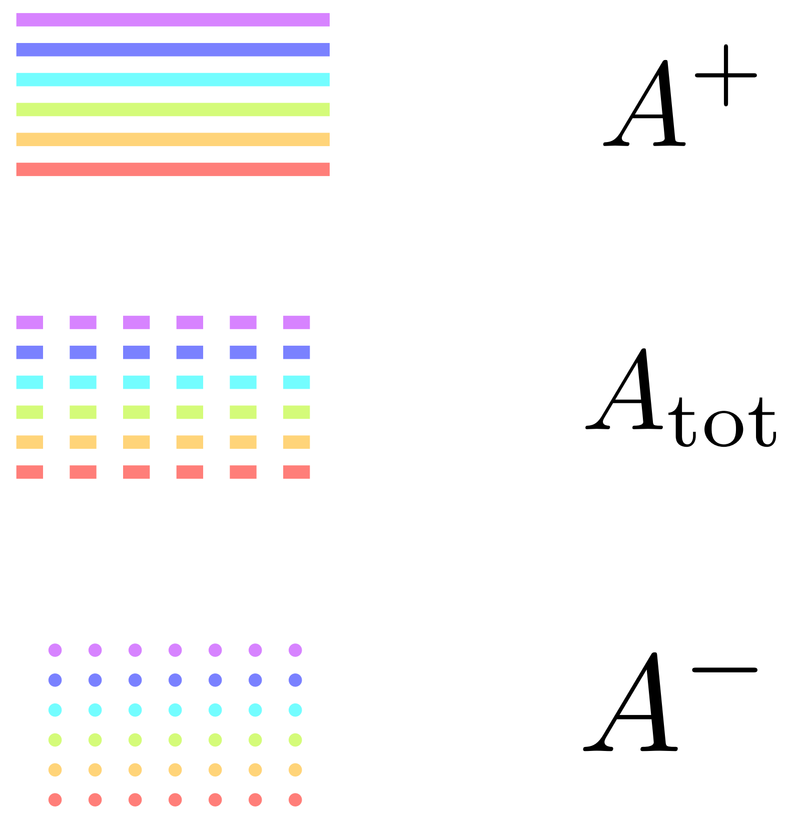
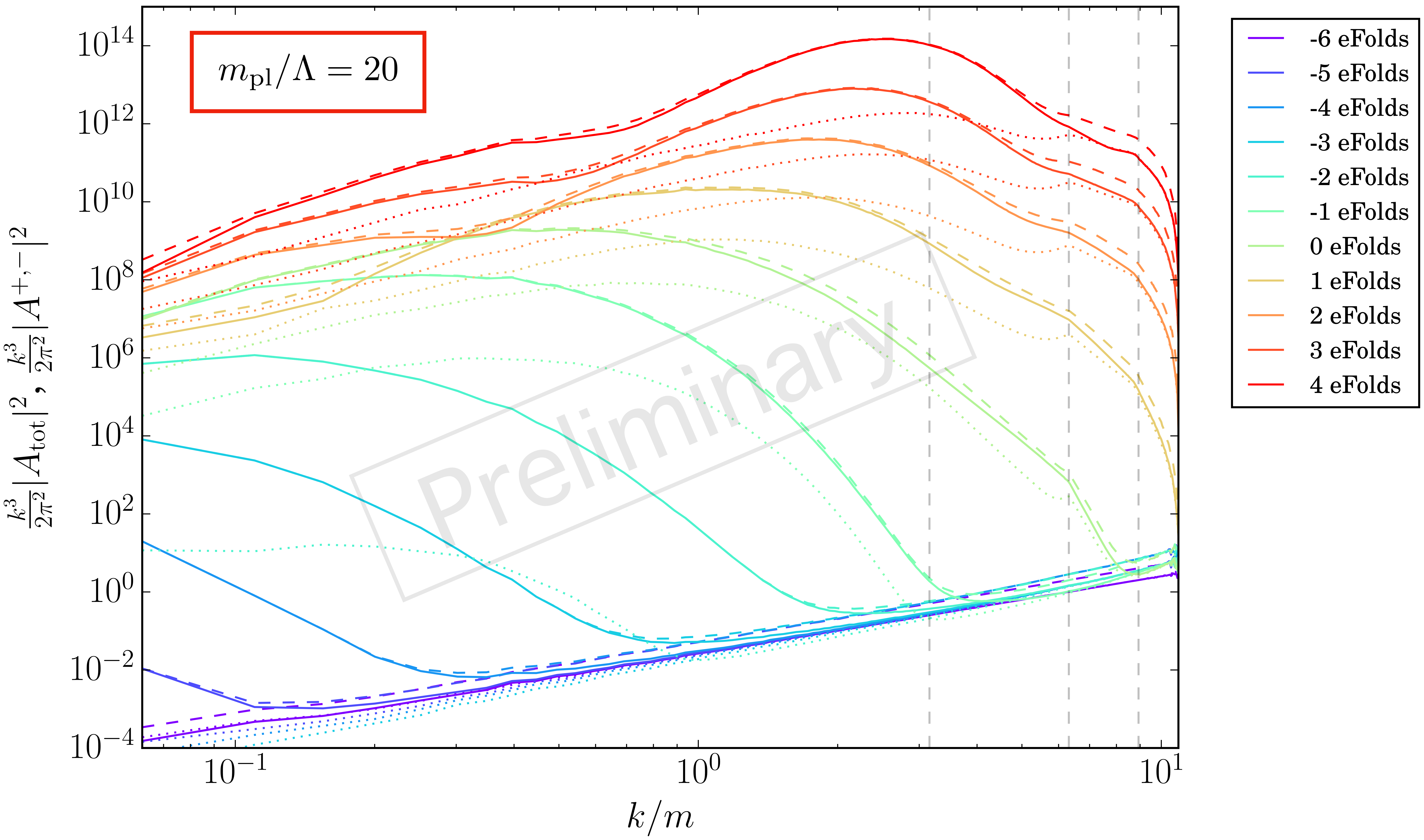
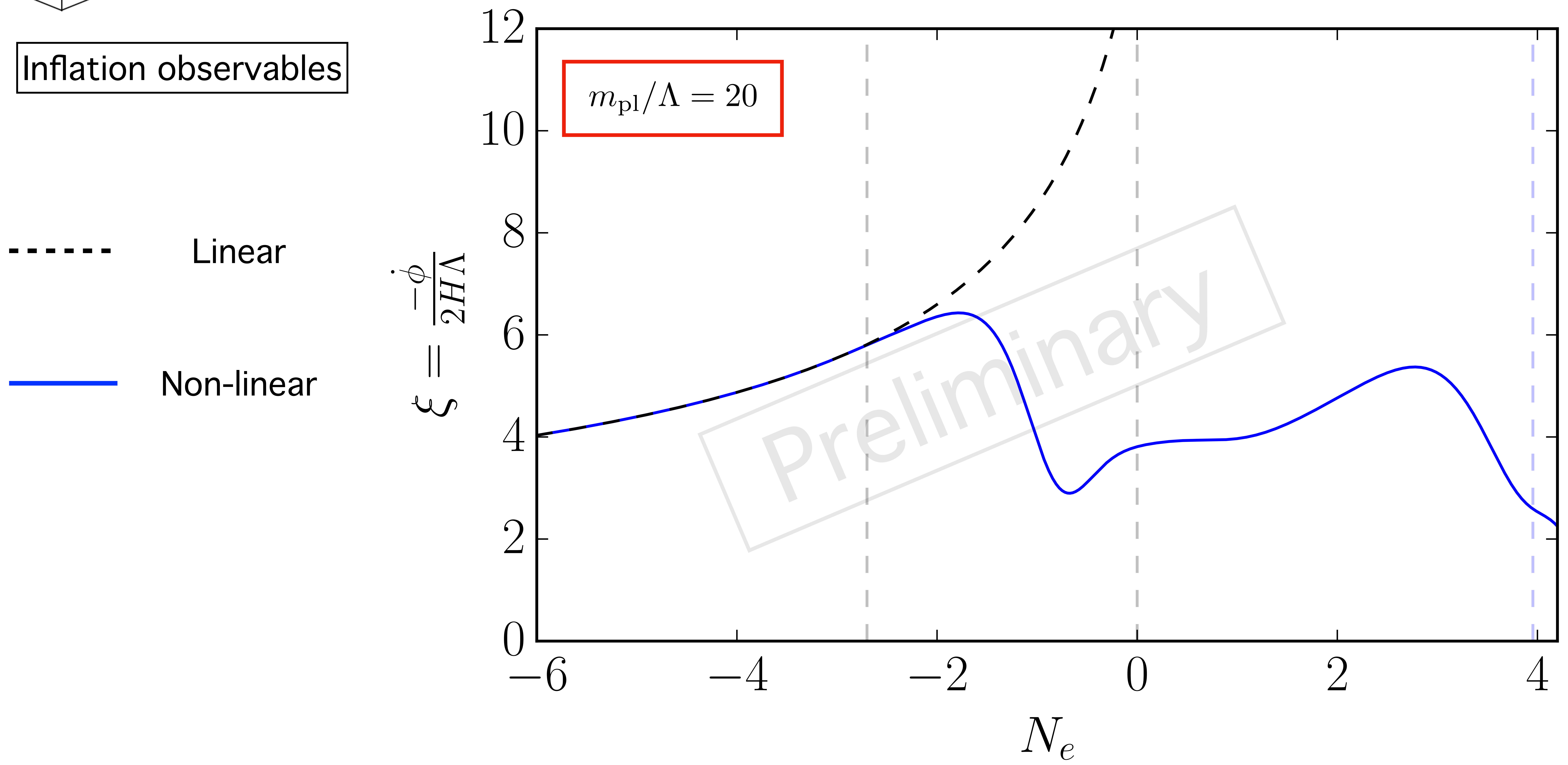




Non-linear evolution: example

Inflation observables



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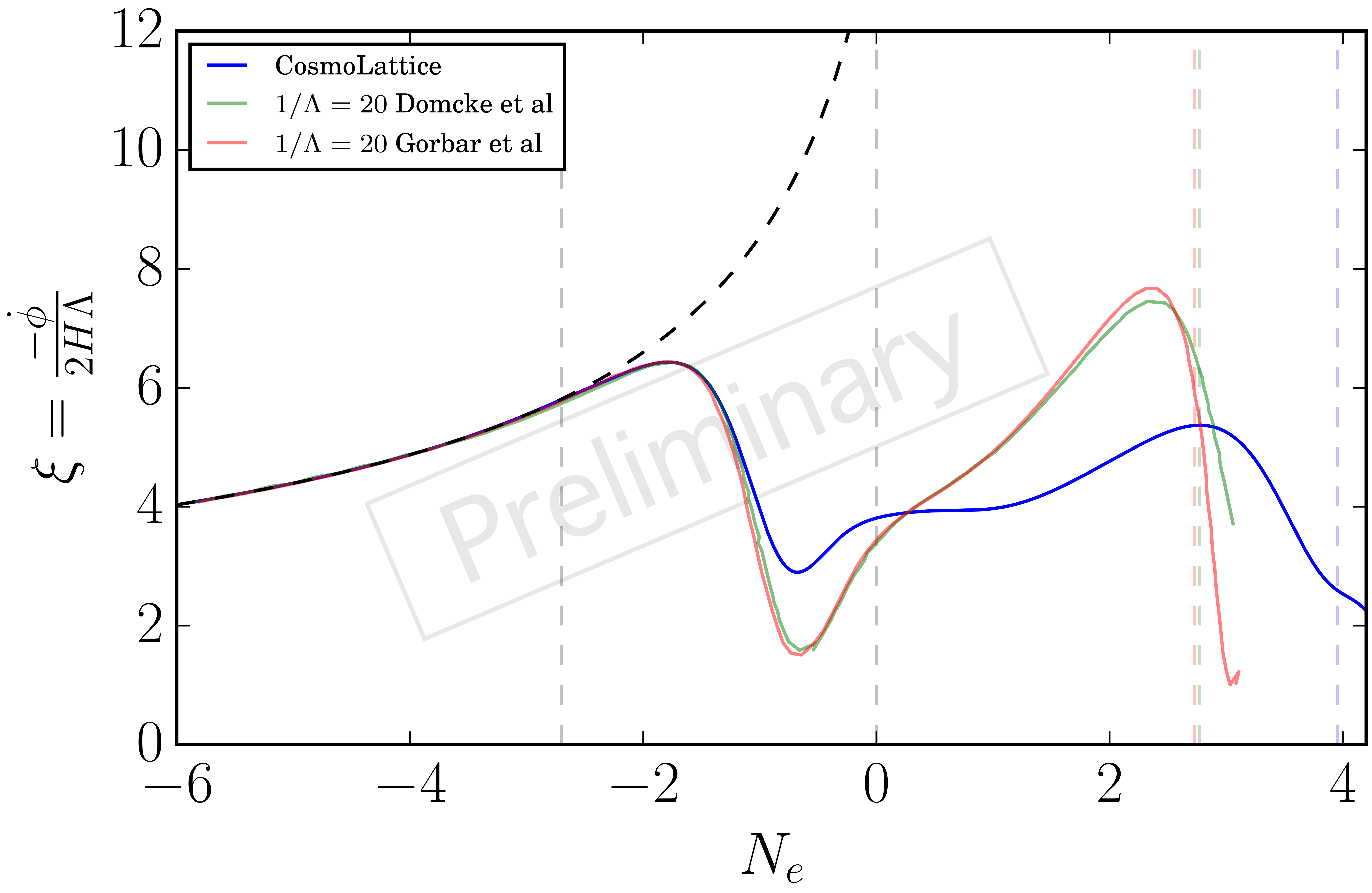
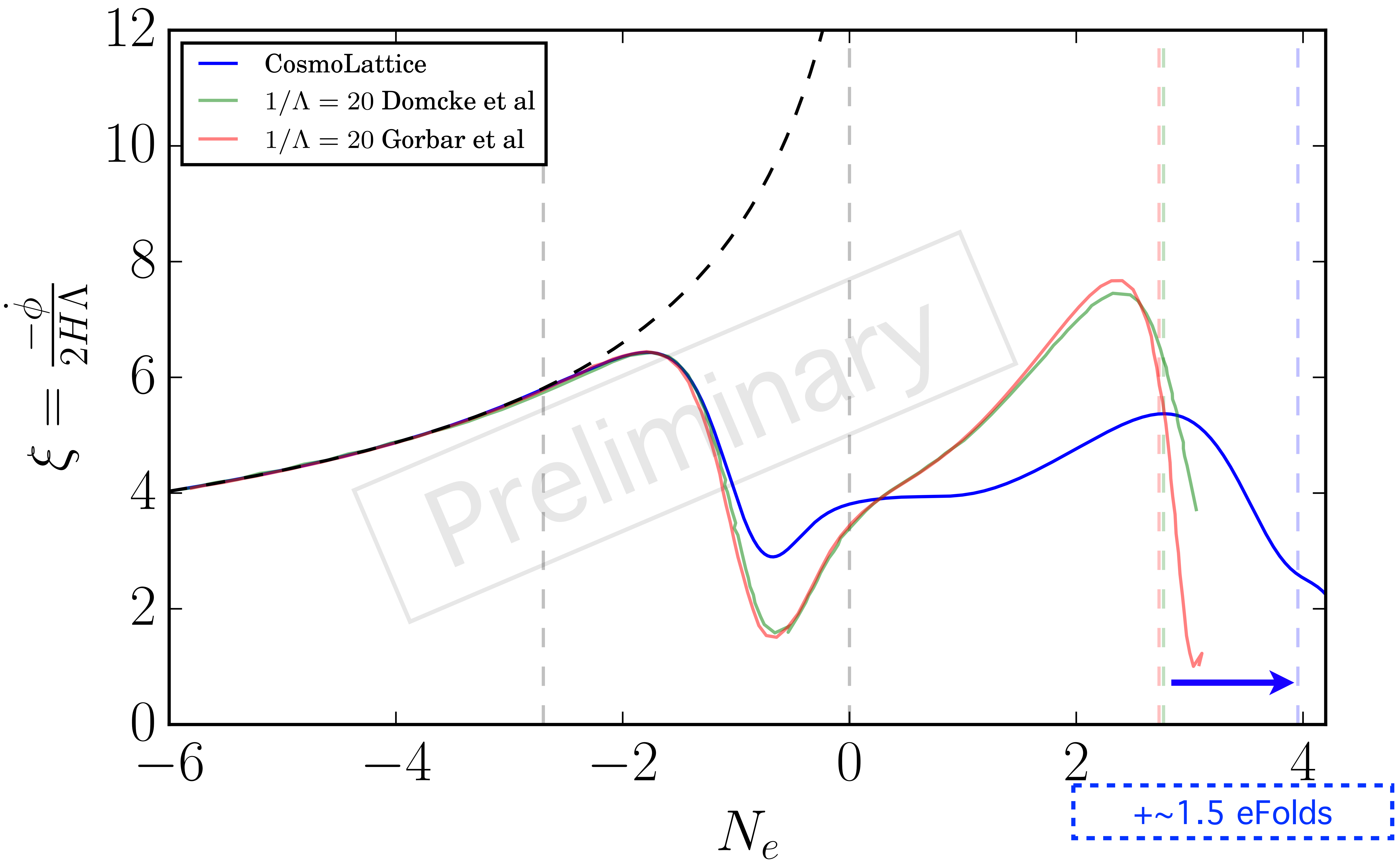


7- Non-linear evolution: example

Inflation observables

Linear

Non-linear



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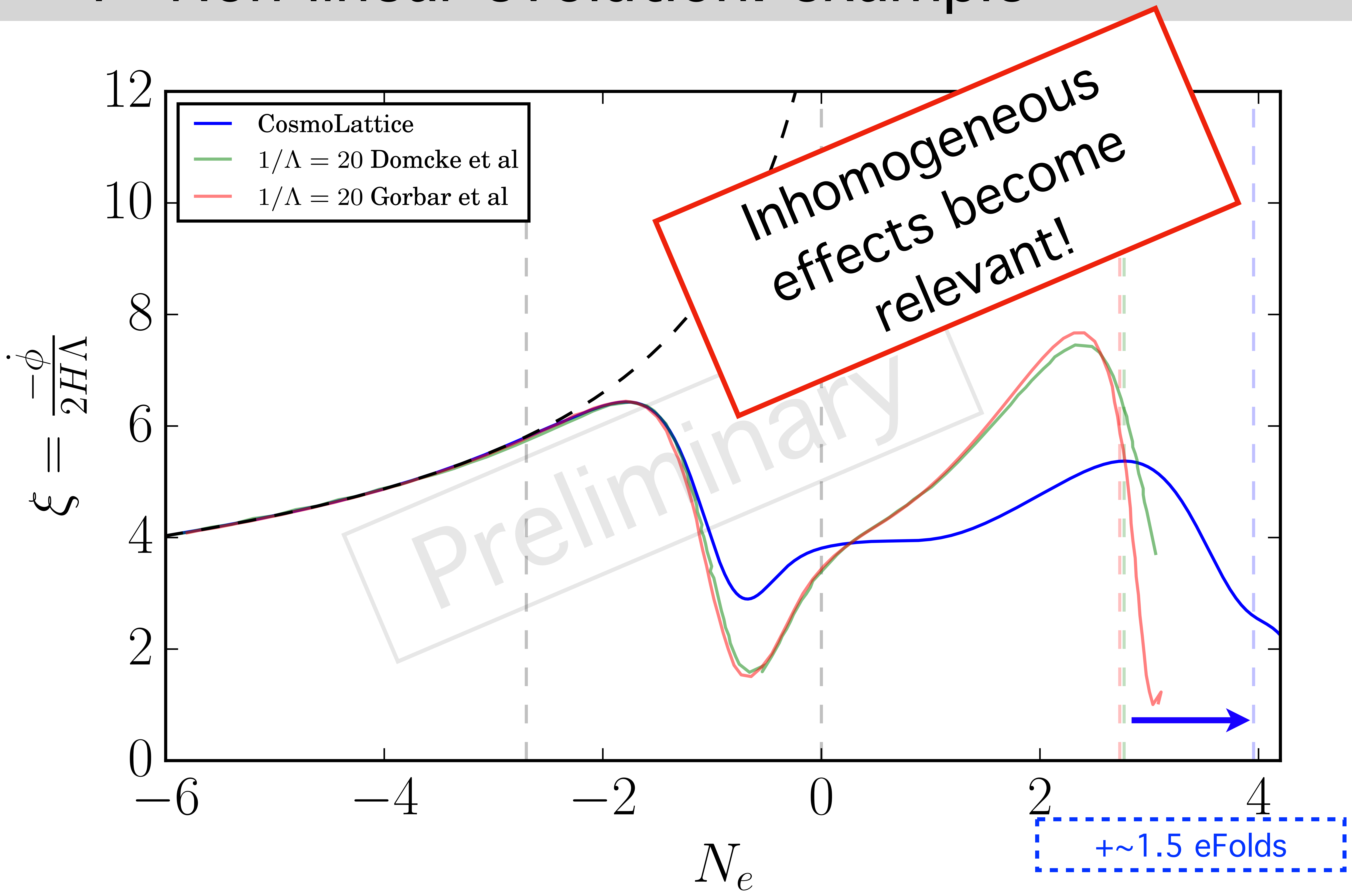

Summary & Conclusions

- Successful implementation of axion-inflation in CL

$$\boxed{\frac{\phi}{\Lambda}F\tilde{F}}$$

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Ready to study non-linear phenomenology!

- Preliminar non-linear sims already show inhomogeneities non-negligible

Thank you!