SU(2) gauge fields

① Continuum:

$$S = -\int dx^4 \; \frac{1}{2} \mathrm{Tr}(G_{\mu\nu} G^{\mu\nu}) + (D_\mu \Phi)^\dagger D^\mu \Phi$$

$$D_\mu \Phi = \partial_\mu \Phi - i g_B Q_B B_\mu \Phi$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i [B_\mu, B_\nu]$$

$$B_\mu, G_{\mu\nu} \in \underbrace{su(2)}_{\text{Lie Algebra}}, \quad \Phi \in \mathbb{C}^2$$

Invariant under local (gauge) $\overset{\text{group}}{SU(2)}$ transfo.

$$B_\mu \to \Omega(x) B_\mu \Omega^\dagger(x) - \frac{i}{g_B Q_B} (\partial_\mu \Omega(x)) \Omega^\dagger(x)$$

$$G_{\mu\nu} \to \Omega(x) G_{\mu\nu} \Omega^\dagger(x)$$

Reminder: $M^\dagger = M$, $\mathrm{tr}(M)$ $\forall M \in su(2)$

$U^\dagger = U^{-1}$, $\det(U) = 1$ $\forall U \in SU(2)$

EoM: $D^\mu G_{\mu\nu} = J_\nu$

$$D^\mu G_{\mu\nu} = \partial^\mu G_{\mu\nu} + i [B^\mu, G_{\mu\nu}]$$

$$J_\mu = 2 g_B Q_B \,\mathrm{Im}(\Phi^\dagger T^a \Phi) T_a$$

$$T_a = \frac{\sigma_a}{2}, \quad \text{gen. of } SU(2).$$

Note: Even $D^\mu G_{\mu\nu} = 0$ admits interacting sol.!

In FLRW:

$$\partial_\mu \to \nabla_\mu$$

$$\nabla_\mu G^{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial (G^{\sigma\nu} \sqrt{g})}{\partial x^\sigma}$$

Exercise: check!

$$\Rightarrow D_0 G_{0i} - a^{-2(1-\alpha)} D_j G_{ji} + (1-\alpha) \frac{a'}{a} G_{0i} = a^{2\alpha} J_i$$

$$D_i G_{0i} = a^2 J_0 \quad \text{Gauss law}$$

② Discretization:

As for $U(1)$: gauge inv $\Leftrightarrow$ link variables

$$U_\mu = e^{-ig_B Q_B \Delta x B_\mu} \in SU(2)$$

$$U_\mu \to \Omega(x)\, U_\mu\, \Omega^\dagger(x+\mu)$$

Non-Abelian: $G_{\mu\nu}$ through links

Reason: $[B_\mu, B_\nu]$

Strategy: Find $O[U] \to \Omega(x)\, O\, \Omega^\dagger(x)$

Notation: $U_{\mu,+\nu} = U_\mu(\vec{x}+\hat{\nu})$

unit vec. in ν dir.

Plaquette : $U_{\mu\nu} = U_\mu\, U_{\nu,+\mu}\, U^\dagger_{\mu,+\nu}\, U^\dagger_\nu$

Continuum limit:

$$U_{ij} \simeq \mathbb{1} - i g_B Q_B \Delta x^2 G_{ij} + \frac{g_B^2 Q_B^2}{2} \Delta x^4 G_{ij} G_{ij} + O(\Delta x^6)$$

Exercise: Show this

$\Rightarrow$ Used to discretize the EoM

$$D_\mu G_{\mu\nu} \simeq \frac{U_{\mu\nu} - U^\dagger_{\nu,-\mu}\, U_{\mu\nu,-\mu}\, U_{\mu,-\mu}}{\Delta x} \equiv D^-_\mu G^{\mu\nu}$$

③ Time evolution:

Set $B_0 = 0 \Leftrightarrow U_0 = \mathbb{1}$

$\Rightarrow D_0 G_{0i} = \partial_0 G_{0i}$

Can use $\Pi_i = a^{1-\alpha} G_{0i}$

$\Rightarrow$

$$\Pi_i' = a^{1+\alpha} J_i + a^{\alpha-1} D^-_\mu G^{\mu\nu} \equiv K_B$$

Apply same algorithms to time evolve!

Subtlety: Drifts.

How to evolve U_i from Π_i?

Solution # 1:

$$(U_i)' = \left(e^{-i g_B Q_B \Delta x B_i}\right)'$$

$$= -i g_B Q_B \Delta x (B_i)' U_i + O(\Delta x^2)$$

$$= \frac{-i g_B Q_B \Delta x}{a^{1-\alpha}} \Pi_i^B U_i + O(\Delta x^2)$$

$\Rightarrow$ Invert this relation for $U_{i,+0}$.

Solution # 2:

ⓐ Compute $U_{0i} = e^{-i g_B Q_B \Delta t \, \Pi_i / a^{1-\alpha}}$

ⓑ $U_{0i} = U_{i,+0} U_i^{\dagger} \Rightarrow U_{i,+0} = U_{0i} U_i$

ⓐ easy for $SU(2)$. Not for $SU(N)$.

Example: Leapfrog scheme.

$$\Pi^B_{i,+\hat{0}/2} = \Pi^B_{i,-\hat{0}/2} + dt\, k_B$$

ⓐ

$$U_{i,+\hat{0}} = U_i - \frac{i g_B Q_B \Delta x \Delta t}{a^{1-\alpha}_{1/2}} \Pi^B_{i,\hat{0}/2} \left(\frac{U_{i+1} + U_i}{2}\right) \quad (*)$$

$(*)$

$$\Rightarrow U_{i+1} = \Pr_{SU(2)} \left(\mathbb{1} - \frac{i g_B Q_B \Delta x \Delta t}{a^{1-\alpha}_{1/2}} \Pi_{i,\hat{0}/2}\right)^2 U_i$$

Note: Gauge inv. disc

$\Updownarrow$

Exact pres. Gauss law

Reason: Gauss law

$\Updownarrow$

Cons. current from shift in A_0

$\equiv$

gauge trans.

Conservation: Analogous to L in one body.

Initial conditions: Similar to $U(1)$.

See CL code