# CosmoLattice – School 2022

# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

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# CosmoLattice – School 2022

**Day 1**
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

(Yesterday)

**Day 2**
(Tuesday 6th)

- Lesson 3: Evolution algorithms ODE - Adrien
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

**Day 3**
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

**Day 4**
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

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- **Practice**

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- **Lesson 3: Evolution algorithms ODE - Adrien** ✓
- **Lesson 4: Interacting scalar fields in an expanding background - Dani**
- **Topical 1: Gravitational non-minimally coupled scalar fields - Ben & Toby**
- **Practice - All**

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# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

# *CosmoLattice* – School 2022

# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

* **L4.a:** $\mathcal{O}(dt^2) - \mathcal{O}(dt^{10})$ **SF@EB Algorithms**
* **L4.b: Models** $\tanh^p(\phi/M)$

# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

* **L4.a:** $\mathcal{O}(dt^2) - \mathcal{O}(dt^{10})$ **SF@EB Algorithms**
* **L4.b: Models** $\tanh^p(\phi/M)$

# *CosmoLattice* – School 2022

## — Lecture 4.a —

$$\mathcal{O}(dt^2) - \mathcal{O}(dt^{10})$$

## SF@EB Algorithms

Scalar Field @ Expanding Background (SF@EB) Algorithms

# Continuum Formulation

**Interacting real scalar fields:** $\{\phi_a\}_{a=1,2,\ldots,N_s}$ **(with canonically normalized kinetic terms)**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Continuum Formulation

**Interacting real scalar fields:** $\{\phi_a\}_{a=1,2,...,N_s}$ **(with canonically normalized kinetic terms)**

**Action:** $$S = -\int d^4x \sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi_b\partial^\mu\phi_b + V(\{\phi_c\})\right) = \left(\frac{f_*}{\omega_*}\right)^2 \tilde{S}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Continuum Formulation

**Interacting real scalar fields:** $\{\phi_a\}_{a=1,2,\ldots,N_s}$ **(with canonically normalized kinetic terms)**

**Action:** $S = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi_b\partial^\mu\phi_b + V(\{\phi_c\})\right) = \left(\frac{f_*}{\omega_*}\right)^2\tilde{S}$

$$\boxed{\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

(FLRW)

$$\left[\begin{array}{c} \tilde{\phi}_a \equiv \dfrac{\phi_a}{f_*}\,, \qquad d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt\,, \qquad d\tilde{x}^i \equiv \omega_* dx^i \\ \widetilde{V}(\{\tilde{\phi}_c\}) \equiv \dfrac{1}{f_*^2\omega_*^2}V(\{\phi_c\})\Big|_{\phi_c = f_*\tilde{\phi}_c} \end{array}\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Continuum Formulation

$$\boxed{\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Continuum Formulation

$$\boxed{\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

$$\delta\tilde{S} = 0 \Rightarrow \boxed{\tilde{\phi}_a'' - a^{-2(1-\alpha)}\tilde{\nabla}^2\tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' + a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_a} = 0}$$

$$\boxed{(a = 1,2,\ldots,N_s)}$$

EOM in the dimensionless variables

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Continuum Formulation

$$\boxed{\tilde{S} = \int d^3\tilde{x}\,d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

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**EOM in the dimensionless variables**

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right],$$

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right],$$

$$\left(\begin{aligned} \widetilde{E}_K &\equiv \frac{1}{2a^{2\alpha}}\sum_i\left\langle(\tilde{\phi}_i')^2\right\rangle \\ \widetilde{E}_G &\equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle \\ \widetilde{E}_V &\equiv \left\langle\tilde{V}(\{\tilde{\phi}_j\})\right\rangle \end{aligned}\right)$$

**EOM of the Expanding Background**

# Lattice Formulation

**Simple Prescription:**

| Continuum | | Lattice |
|---|---|---|
| $\partial_\mu$ | $\longrightarrow$ | $\nabla_\mu^{(\mathrm{L})}$ |

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice Formulation

**Simple Prescription:**

| Continuum | | Lattice |
| --- | --- | --- |
| $\partial_\mu$ | $\longrightarrow$ | $\nabla_\mu^{(\mathrm{L})}$ |

**Valid for Scalar Fields** ✓

↓

**Not enough for Gauge theories !**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^{(\mathrm{L})}_\mu \end{pmatrix}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

**i) EOM discretization approach**: Discretize the continuum EOM

**ii) Lattice action approach**: Discretize the continuum action

# Scalar Field @ Expanding Background (SF@EB) Algorithms

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**i) EOM discretization approach**: Discretize the continuum EOM

$$\tilde{\phi}_a'' - a^{-2(1-\alpha)}\tilde{\nabla}^2\tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' = -a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_a} \quad \Longrightarrow \quad \text{Discretize:} \quad \partial_\mu \longrightarrow \nabla_\mu^{(\mathrm{L})}$$

**ii) Lattice action approach**: Discretize the continuum action

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**ii) Lattice action approach**: Discretize the continuum action

$$\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Discretize: $\partial_\mu \longrightarrow \nabla_\mu^{(\mathrm{L})}$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice Formulation

$$\left(\begin{array}{ccc} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{array}\right)$$

**i) EOM discretization approach**: Discretize the continuum EOM

$$\tilde{\phi}_a'' - a^{-2(1-\alpha)}\tilde{\nabla}^2\tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' = -\,a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_a}$$

**ii) Lattice action approach**: Discretize the continuum action

$$\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

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**i) EOM discretization approach**: Discretize the continuum EOM

$$\tilde{\phi}_a'' - a^{-2(1-\alpha)}\tilde{\nabla}^2\tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' = -a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_a}$$

?

**ii) Lattice action approach**: Discretize the continuum action

$$\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x} d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b^{\prime 2} - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

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**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

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**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \underbrace{\sum_{n_0} \sum_{\mathbf{n}}} \left\{ \frac{1}{2} a^{3-\alpha}_{+0/2} \sum_b (\widetilde{\Delta_0^+} \tilde{\phi}^b)^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta_i^+} \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

time step

lattice spacing

integration $\simeq$ Sum

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^{(\mathrm{L})}_\mu \end{pmatrix}$$

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**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \sum_{n_0} \sum_{\mathbf{n}} \left\{ \frac{1}{2} a^{3-\alpha}_{+0/2} \sum_b (\widetilde{\Delta}_0^+ \tilde{\phi}^b)^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta}_i^+ \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Forward
Derivatives

# Scalar Field @ Expanding Background (SF@EB) Algorithms

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**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}^+_0\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}^+_i\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

scale factor at $(n_0 + 1/2)\delta\tilde{\eta}$

**(semi-integer times)**

scale factor at $n_0\delta\tilde{\eta}$

**(integer times)**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

$$\underbrace{\frac{\tilde{\phi}^b(n_0+1)-\tilde{\phi}^b(n_0)}{\delta\tilde{\eta}}}_{\equiv (\widetilde{\Delta}_0^+\tilde{\phi}^b)\big|_{\mathbf{n},n_0+1/2}} \qquad \underbrace{\frac{\tilde{\phi}^b(\mathbf{n}+\hat{i})-\tilde{\phi}^b(\mathbf{n})}{\delta\tilde{x}^i}}_{\equiv (\widetilde{\Delta}_i^+\tilde{\phi}^b)\big|_{\mathbf{n}+\hat{i}/2,n_0}}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

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$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

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**(semi-integer times)** **(integer times)**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^{(\mathrm{L})}_\mu \end{pmatrix}$$

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$$\delta_{\phi_a}S_{\rm L} = 0 \quad \Rightarrow \quad \widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

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$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a_{+0/2}^{3-\alpha}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}\ \text{(semi-integer times)}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}\ \text{(integer times)}}\right\}$$

$$\delta_{\phi_a}S_{\rm L} = 0 \quad \Rightarrow \quad \underbrace{\widetilde{\Delta}_0^-}_{\text{Backward time deriv.}}[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \underbrace{\widetilde{\Delta}_i^-}_{\text{Backward spat. deriv.}}\ \text{x}\ \underbrace{\widetilde{\Delta}_i^+}_{\text{Backward spat. deriv.}}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta} \left\{ \frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3 \sum_{n_0}\sum_{\mathbf{n}} \left\{ \underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b (\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{\substack{@\ (n_0+1/2)\delta\tilde{\eta} \\ \textbf{(semi-integer times)}}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{\substack{@\ n_0\delta\tilde{\eta} \\ \textbf{(integer times)}}} \right\}$$

$$\delta_{\phi_a}S_{\rm L} = 0 \;\Longrightarrow\; \widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

scale factor
at $(n_0+1/2)\delta\tilde{\eta}$
**(semi-integer times)**

scale factor
at $n_0\delta\tilde{\eta}$
**(integer times)**

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$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

**(semi-integer times)** **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \quad \Rightarrow \quad \underbrace{\widetilde{\Delta}_0^-[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b}_{@\ (n_0+1/2)\delta\tilde{\eta}}]}_{@\ n_0\delta\tilde{\eta}} = \underbrace{a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{@\ n_0\delta\tilde{\eta}}\,; \qquad b = 1,2,\ldots,N_s$$

**(integer times)** **(integer times)**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \sum_{n_0} \sum_{\mathbf{n}} \left\{ \underbrace{\frac{1}{2} a^{3-\alpha}_{+0/2} \sum_b (\widetilde{\Delta}_0^+ \tilde{\phi}^b)^2}_{\substack{@\ (n_0+1/2)\delta\tilde{\eta} \\ \textbf{(semi-integer times)}}} - \underbrace{\frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta}_i^+ \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\})}_{\substack{@\ n_0\delta\tilde{\eta} \\ \textbf{(integer times)}}} \right\}$$

$$\delta_{\phi_a} S_{\rm L} = 0 \quad \Rightarrow \quad \underbrace{\widetilde{\Delta}_0^- [\underbrace{a^{3-\alpha}_{+0/2} \widetilde{\Delta}_0^+ \tilde{\phi}_b}_{@\ (n_0+1/2)\delta\tilde{\eta}}]}_{\substack{@\ n_0\delta\tilde{\eta} \\ \textbf{(integer times)}}} = \underbrace{a^{1+\alpha} \sum_i \widetilde{\Delta}_i^- \widetilde{\Delta}_i^+ \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b}}_{\substack{@\ n_0\delta\tilde{\eta} \\ \textbf{(integer times)}}}\ ; \qquad b = 1,2,\ldots,N_s$$

**LHS** $\Longleftrightarrow$ **RHS**

consistency ✓

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^{(\mathrm{L})}_\mu \end{pmatrix}$$

**Continuum:**

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}'^2_b - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

**Lattice:**

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

**(semi-integer times)** **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \;\Rightarrow\; \boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,;} \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,;} \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,;} \quad b = 1,2,\ldots,N_s$$

**Where does the scale factor live ?**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:**

$$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta_0^+\tilde{\phi}_b})^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta_k^+\tilde{\phi}_b})^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

**[semi-integer times]** **[integer times]** **[integer times]**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\left(\begin{array}{ccc}\text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)}\end{array}\right)$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

**i) IF** $a(n_0 + 1/2)$

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle (\widetilde{\Delta}_0^+\tilde{\phi}_b)^2 \right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle (\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

**[semi-integer times]** **[integer times]** **[integer times]**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

**Cont.** **Lattice**

**i) IF** $a(n_0 + 1/2) \quad \Longrightarrow \quad \begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,, \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,;\quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

**Cont.** **Lattice**

**i) IF** $a(n_0+1/2)$ $\Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

**[semi-integer times]**

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b\left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

**[semi-integer times]** **[integer times]** **[integer times]**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

**Cont.** **Lattice**

**i) IF** $a(n_0+1/2)$ $\Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

**[semi-integer times]**

$$\overline{\widetilde{E}}_G = \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \overline{\widetilde{E}}_V = \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

consistency ✓

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$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

**[semi-integer times]** **[integer times]** **[integer times]**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

### Where does the scale factor live ?

**Cont.** **Lattice**

**i) IF** $a(n_0+1/2) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

**[semi-integer times]**

$$\overline{\widetilde{E}}_G \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \overline{\widetilde{E}}_V \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

**ii) IF** $a(n_0) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

| | **Cont.** | | **Lattice** |
|---|---|---|---|

**i) IF** $a(n_0+1/2) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

**[semi-integer times]**

$$\overline{\widetilde{E}}_G \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \overline{\widetilde{E}}_V \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

**ii) IF** $a(n_0) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\text{, [semi-integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

**[integer times]**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

**Cont:** $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

**Where does the scale factor live ?**

**Cont.** **Lattice**

**i) IF** $a(n_0 + 1/2) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

**[semi-integer times]**

$$\overline{\widetilde{E}}_G \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}, \qquad \overline{\widetilde{E}}_V \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

**ii) IF** $a(n_0) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2} \text{ , [semi-integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

**[integer times]**

consistency ✔

$$\overline{\widetilde{E}}_K \to \frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action/EOM approach

$$\left(\begin{array}{ccc}\text{Continuum} & & \text{Lattice}\\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)}\end{array}\right)$$

**Scalar field dynamics**

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,;}\qquad b = 1,2,\ldots,N_s$$

**Expansion of the Universe**

**i) IF** $a(n_0+1/2)$ $\Longrightarrow$

$$\left\{\begin{array}{l}\widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,,\\[2mm] \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\Big[(\alpha-2)\widetilde{E}_K + \alpha\,\overline{\widetilde{E}_G} + (\alpha+1)\,\overline{\widetilde{E}_V}\Big]\end{array}\right.$$

$$\overline{\widetilde{E}_G} = \frac{(\widetilde{E}_G+\widetilde{E}_{G,+\hat{0}})}{2}\,,\qquad \overline{\widetilde{E}_V} = \frac{(\widetilde{E}_V+\widetilde{E}_{V,+\hat{0}})}{2}$$

**ii) IF** $a(n_0)$ $\Longrightarrow$

$$\left\{\begin{array}{l}\widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,,\\[2mm] \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\Big[(\alpha-2)\,\overline{\widetilde{E}_K} + \alpha\,\widetilde{E}_G + (\alpha+1)\,\widetilde{E}_V\Big]\end{array}\right.$$

$$\overline{\widetilde{E}_K} = \frac{(\widetilde{E}_{K,-\hat{0}/2}+\widetilde{E}_{K,+\hat{0}/2})}{2}$$

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Lattice action/EOM approach

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(+)} \end{pmatrix}$$

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$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,;} \qquad b = 1,2,\ldots,N_s$$

**Expansion of the Universe**

**i) IF** $a(n_0 + 1/2)$ $\Longrightarrow$

$$\begin{cases} \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,, \\ \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}$$

$$\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

**ii) IF** $a(n_0)$ $\Longrightarrow$

$$\begin{cases} \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

# How do we solve these EOM ? …

# Can we apply previous evolution algorithms ?

* LeapFrog
* Verlet methods
* Runge-Kutta methods
* Higher Order integrators

**Notation**

$$\phi_{+0} \equiv \phi(\mathbf{n}, n_0)$$

$$\pi_{+0/2} \equiv \pi(\mathbf{n}, n_0 + 1/2)$$

# Can we apply previous evolution algorithms ?

* LeapFrog
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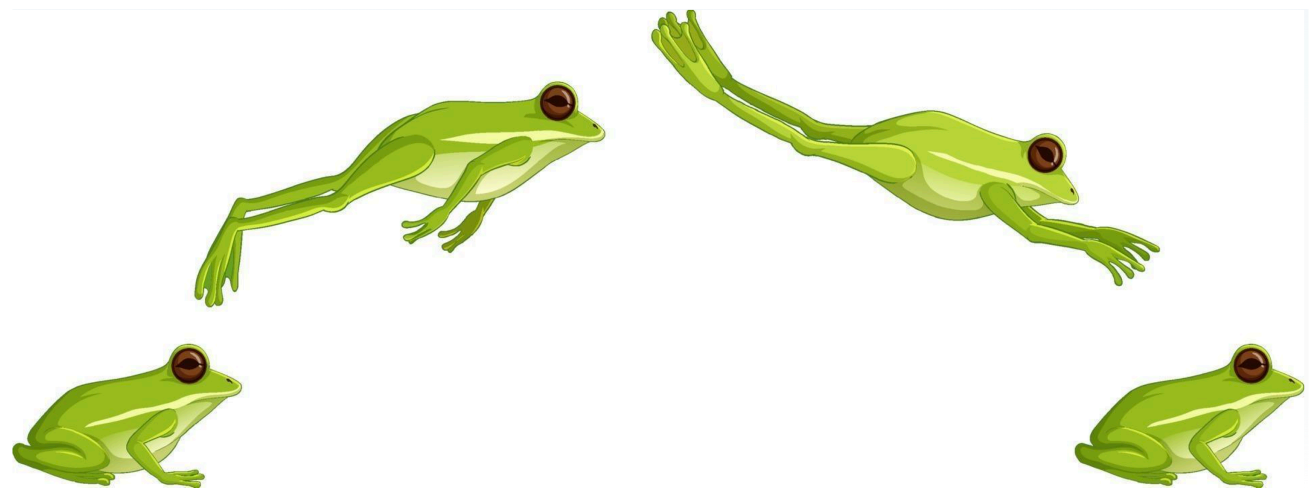
**Notation**

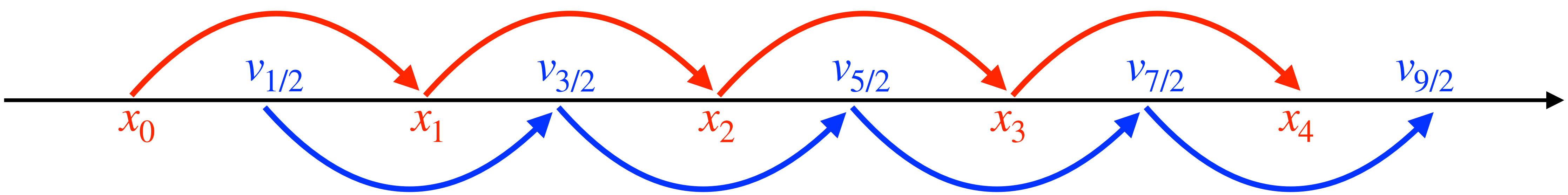
$$\phi_{+0} \equiv \phi(\mathbf{n}, n_0)$$

$$\pi_{+0/2} \equiv \pi(\mathbf{n}, n_0 + 1/2)$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog





# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

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## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

**EOM:** $$\widetilde{\Delta}^{-}_{0}[a^{3-\alpha}_{+0/2}\underbrace{\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b}_{\equiv\, \tilde{\pi}^{(b)}_{+0/2}}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

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**EOM:** $$\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\underbrace{\widetilde{\Delta}_0^+\tilde{\phi}_b}_{\equiv \tilde{\pi}^{(b)}_{+0/2}}] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

---

$$\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

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*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}^+_0 \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$IC \quad : \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

---

$$\widetilde{\Delta}^-_0[a^{3-\alpha}_{+0/2}\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^-_i \widetilde{\Delta}^+_i \tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

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## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$\begin{aligned} IC &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\ a_{+0/2} &= a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \end{aligned}$$

---

$$\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,...,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$
\begin{aligned}
IC \quad &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} \quad &= \quad a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} \quad &= \quad \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} \; + \; a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta}
\end{aligned}
$$

---

$$
\widetilde{\Delta}^{-}_{0}[a^{3-\alpha}_{+0/2}\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

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*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$\begin{aligned}
IC \quad &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} &= a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} &= \underbrace{\left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} + a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}_k^- \widetilde{\Delta}_k^+ \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta}}_{\tilde{\pi}^{(a)}_{-0/2} + \mathscr{K}_{\tilde{\phi}_a}[\{\tilde{\phi}^{(b)}, a_{-0/2}, a, a_{+0/2}, \tilde{\pi}^{(a)}_{-0/2}]\delta\tilde{\eta}}
\end{aligned}$$

**Non-conservative force**

**(no good for symplectic integrator)**

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$$\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}_i^- \widetilde{\Delta}_i^+ \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*II) Iterative scheme for* $\tilde{\pi}^{(a)} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_{-0/2}^{(a)}$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{a}, \tilde{\pi}^{(a)}\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\phi}_{-0/2}^{(a)}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*II) Iterative scheme for* $\tilde{\pi}^{(a)} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_{-0/2}^{(a)}$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{a}, \tilde{\pi}^{(a)}\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\phi}_{-0/2}^{(a)}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

.

.

.

$$\tilde{\pi}_{+0}^{(a)} = \tilde{\pi}^{(a)} + \mathscr{K}_{\tilde{\phi}_a}[\{\tilde{\phi}_{+0/2}^{(b)}\}, a, a_{+0/2}, a_{+0}, \tilde{\pi}^{(a)}]\delta\tilde{\eta}$$

Non-conservative force

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

**EOM:** $$\widetilde{\Delta}^{-}_{0}[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b}_{\equiv\, \tilde{\pi}^{(b)}_{+0/2}}] = \underbrace{a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{\mathscr{K}_{\tilde{\phi}_a}[\{\tilde{\phi}^{(b)}\},\, a]}\,; \qquad b = 1,2,\ldots,N_s$$

**Conservative force !**

**(Appropriate for symplectic integrators)**

---

$$\widetilde{\Delta}^{-}_{0}[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

---

$$\widetilde{\Delta}^{-}_{0}[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a, \} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\left[ \quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}} \sum_b \left\langle (\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2 \right\rangle , \quad \widetilde{E}_G \equiv \frac{1}{2a^2} \sum_{b,k} \left\langle (\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2 \right\rangle , \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad \right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a,\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} = \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\ b_{+0/2} = b_{-0/2} + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^+_0\tilde{\phi}_a$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a,\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} = \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\ b_{+0/2} = b_{-0/2} + \dfrac{\delta\tilde{\eta}}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\begin{cases} a_{+0} = a + b_{+0/2}\,\delta\tilde{\eta} \quad \longrightarrow \quad a_{+0/2} \equiv (a_{+0} + a_0)/2\,, \\ \tilde{\phi}^{(a)}_{+0} = \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*III) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a, \} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} = \tilde{\pi}^{(a)}_{-0/2} + \left( a^{1+\alpha} \sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta} \\ b_{+0/2} = b_{-0/2} + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\begin{cases} a_{+0} = a + b_{+0/2}\,\delta\tilde{\eta} \quad \longrightarrow \quad a_{+0/2} \equiv (a_{+0} + a_0)/2\,, \\ \tilde{\phi}^{(a)}_{+0} = \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2} a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

$$HC \quad : \quad b^2_{+0/2} = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}_{+0/2}\left(\widetilde{E}_K + \overline{\widetilde{E}_G} + \overline{\widetilde{E}_V}\right),$$

---

$$\left[ \quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}} \sum_b \left\langle (\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2} \sum_{b,k} \left\langle (\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad \right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*IV) Iterative scheme for* $\tilde{\pi}^{(a)} \equiv a^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}^{(a)}_{-0/2}$ and scale factor $a(n_0+1/2)$

**Pretty much the same as** *III)*

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle (\widetilde{\Delta}_0^+\tilde{\phi}_b)^2 \right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle (\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

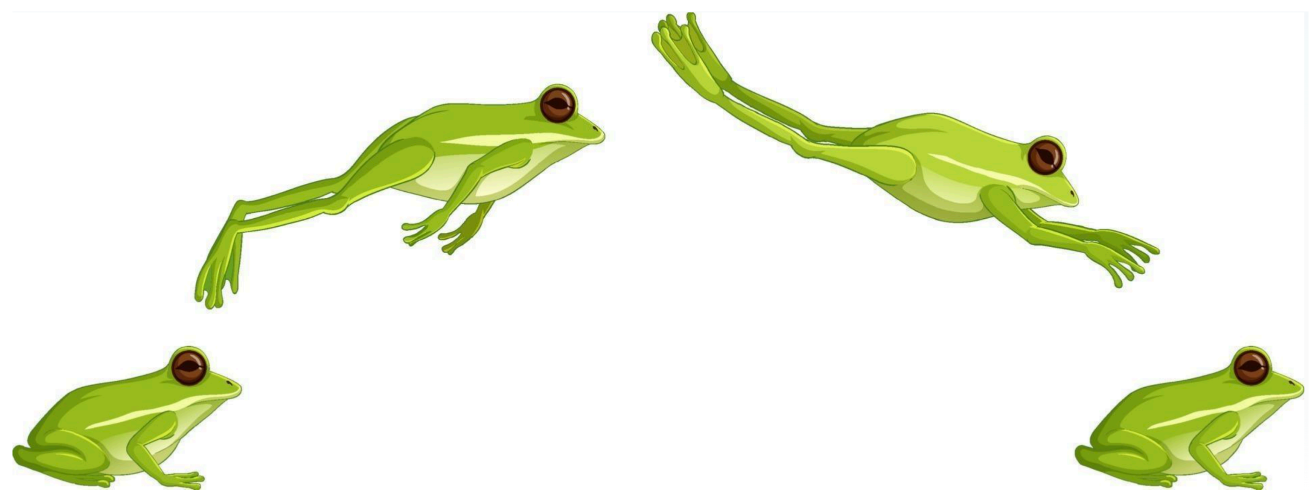
## (Staggered) LeapFrog

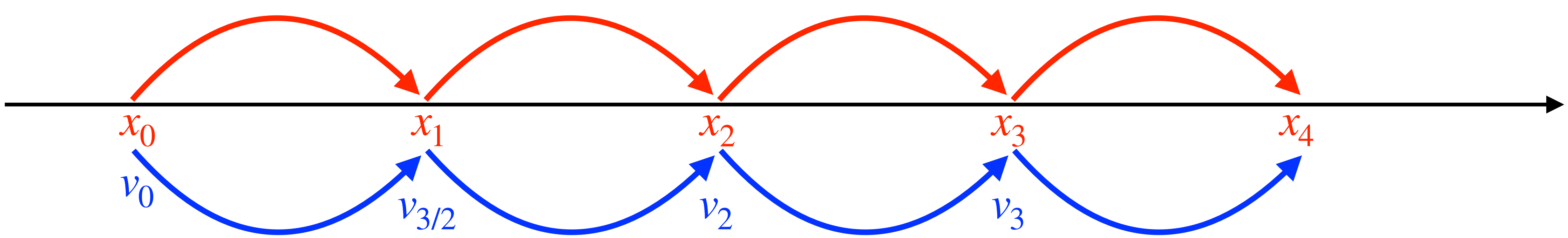
**"The Art" (2006.15122) presented 4 LF methods:**

| Label | Discussed here | Order | CosmoLattice | |
|---|---|---|---|---|
| I | ✓ | $\mathcal{O}(dt)$ | ✗ | (Non-conservative) |
| II | ✗ | $\mathcal{O}(dt)$ | ✗ | (~~Symplectic integ.~~) |
| III | ✓ | $\mathcal{O}(dt^2)$ | ✓ | (Conservative) |
| IV | ✗ | $\mathcal{O}(dt^2)$ | ✗ | (Symplectic integ.) |

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Synchronized LeapFrog

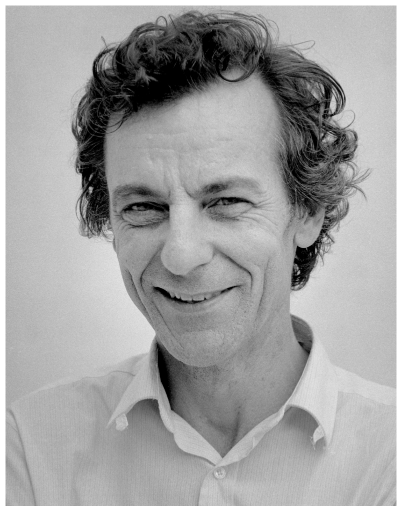




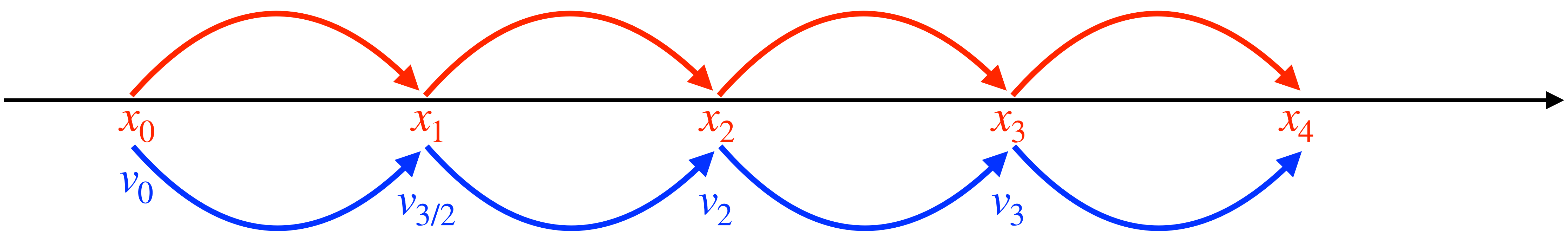
# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

Loup Verlet



**RIP 13th/06/2019**



# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

**Continuum:**

**Fld dynamics:**

$$(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0\,, \qquad i = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

**Continuum:**

**Fld dynamics:**

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,\ldots,N_s$$

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

**Continuum:**

**Fld dynamics:**

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,...,N_s$$

$$\left\{\begin{array}{l} \tilde{\phi}_i' = \overbrace{a^{-(3-\alpha)}\tilde{\pi}_i}^{\mathscr{D}_{\tilde{\phi}_i}(\tilde{\pi}_i,\, a)\ \textbf{(Drift)}}\,, \\ \tilde{\pi}_i' = \underbrace{a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}}_{\textbf{(Kernel)}\ \mathscr{K}_{\phi_i}(\{\phi_j\},\, a)} \end{array}\right\}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

**Continuum:**

**Fld dynamics:**

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,\ldots,N_s$$

$$\left\{\begin{array}{l} \tilde{\phi}_i' = \overbrace{a^{-(3-\alpha)}\tilde{\pi}_i}^{\mathscr{D}_{\tilde{\phi}_i}(\tilde{\pi}_i, a)\ \textbf{(Drift)}}, \\ \tilde{\pi}_i' = \underbrace{a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}}_{\textbf{(Kernel)}\ \mathscr{K}_{\phi_i}(\{\phi_j\}, a)} \end{array}\right\} \Rightarrow$$

**Conservative force**

# Scalar Field @ Expanding Background (SF@EB) Algorithms

# Verlet Algorithms

**Continuum:**

**Fld dynamics** $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms

**Continuum:**

**Fld dynamics** $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i \,, \\ \tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

**Exp. Background:**

$$\begin{cases} a' = b\,, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\sum_i \langle(\tilde{\pi}_i)^2\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_j\})\right\rangle\,, \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms (two flavours)

**position-Verlet** | **velocity-Verlet**

**Continuum:**

**Fld dynamics** $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i \,, \\ \tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

**Exp. Background:**

$$\begin{cases} a' = b \,, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] , \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i \left\langle(\tilde{\pi}_i)^2\right\rangle \,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle \,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_j\})\right\rangle \,, \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Algorithms
## (two flavours)

**position-Verlet** **velocity-Verlet**

**Continuum:**

**Fld dynamics** $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\,\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{\tilde{\phi}} \end{cases} \qquad i = 1,2,\ldots,N$$

**Lattice EOM Approach**

**Exp. E**

$$\begin{cases} a' = b\,, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i \langle(\tilde{\pi}_i)^2\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_j\})\right\rangle\,, \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Velocity-Verlet Integration

*I) Velocity-Verlet scheme for interacting scalar fields in an expanding background*

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0 \,,$$

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$$\left.\begin{cases} b_{+0/2} = b + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]\dfrac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}^{(i)}_{+0/2} = \tilde{\pi}^{(i)} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(i)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(i)}}\right)\dfrac{\delta\tilde{\eta}}{2}\,, \end{cases}\right\} \quad \text{1/2 - step}$$

---

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad \qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,,$$

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$$\left.\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde{\eta} \,, \\ \left[a_{+0/2} = \frac{a_{+0} + a}{2} \,,\right] \\ \tilde{\phi}^{(i)}_{+0} = \tilde{\phi}^{(i)} + \delta\tilde{\eta}\,\tilde{\pi}^{(i)}_{+0/2} a^{-(3-\alpha)}_{+0/2} \,, \end{cases}\right\} \quad \text{Full - step}$$

---

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i \,,$$

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$$\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde\eta\,, \\ \left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\ \tilde\phi^{(i)}_{+0} = \tilde\phi^{(i)} + \delta\tilde\eta\,\tilde\pi^{(i)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

$$\left.\begin{cases} \tilde\pi^{(i)}_{+0} = \tilde\pi^{(i)}_{+0/2} + \left(a^{1+\alpha}_{+0}\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde\phi^{(i)}_{+0} - a^{3+\alpha}_{+0}\widetilde{V}_{,\tilde\phi^{(i)}}\Big|_{+0}\right)\dfrac{\delta\tilde\eta}{2}\,, \\ b_{+0} = b_{+0/2} + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0}\left[(\alpha-2)\widetilde{E}_{K,+0} + \alpha\widetilde{E}_{G,+0} + (\alpha+1)\widetilde{E}_{V,+0}\right]\dfrac{\delta\tilde\eta}{2}\,, \end{cases}\right\}\ \text{1/2 - step}$$

---

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde\phi_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde\phi_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde V(\{\tilde\phi_b\})\right\rangle\ \right]$$

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$$\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde{\eta}\,, \\ \left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\ \tilde{\phi}^{(i)}_{+0} = \tilde{\phi}^{(i)} + \delta\tilde{\eta}\,\tilde{\pi}^{(i)}_{+0/2}a_{+0/2}^{-(3-\alpha)}\,, \end{cases}$$

$$\begin{cases} \tilde{\pi}^{(i)}_{+0} = \tilde{\pi}^{(i)}_{+0/2} + \left(a_{+0}^{1+\alpha}\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(i)}_{+0} - a_{+0}^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(i)}}\Big|_{+0}\right)\dfrac{\delta\tilde{\eta}}{2}\,, \\ b_{+0} = b_{+0/2} + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_{K,+0} + \alpha\widetilde{E}_{G,+0} + (\alpha+1)\widetilde{E}_{V,+0}\right]\dfrac{\delta\tilde{\eta}}{2}\,, \end{cases}$$

$$HC \quad : \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left(\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right)\,,$$

---

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Position-Verlet Integration

*II) Position-Verlet scheme for interacting scalar fields in an expanding background*

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0 \,,$$

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i \,,$$

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---

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---

$$\tilde\pi_i' = a^{1+\alpha}\tilde\nabla^2\tilde\phi_i - a^{3+\alpha}\tilde V_{,\tilde\phi_i} \qquad\qquad \tilde\phi_i' = a^{-(3-\alpha)}\tilde\pi_i\,,$$

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$$\tilde\pi_i' = a^{1+\alpha}\tilde\nabla^2\tilde\phi_i - a^{3+\alpha}\tilde V_{,\tilde\phi_i} \qquad\qquad \tilde\phi_i' = a^{-(3-\alpha)}\tilde\pi_i\,,$$

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$$
\begin{aligned}
IC \quad &: \quad \{\tilde\phi^{(i)}, \tilde\pi^{(i)}, a, b\} \text{ at } \tilde\eta_0\,,\\
a_{+0/2} &= a + b\,\frac{\delta\tilde\eta}{2}\,,\\
\tilde\phi^{(i)}_{+0/2} &= \tilde\phi^{(i)} + \frac{\delta\tilde\eta}{2}\tilde\pi^{(i)}a^{-(3-\alpha)}\,,\\
\tilde\pi^{(i)}_{+0} &= \tilde\pi^{(i)} + \left(a^{1+\alpha}_{+0/2}\sum_k \widetilde\Delta^-_k\widetilde\Delta^+_k\tilde\phi^{(i)}_{+0/2} - a^{3+\alpha}_{+0/2}\widetilde V_{,\tilde\phi^{(i)}}\Big|_{+0/2}\right)\delta\tilde\eta\,,\\
b_{+0} &= b + \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\Big[(\alpha-2)\overline{\widetilde E_K} + \alpha\widetilde E_G + (\alpha+1)\widetilde E_V\Big]\delta\tilde\eta\,,\\
a_{+0} &= a_{+0/2} + b_{+0}\,\frac{\delta\tilde\eta}{2}\,,\\
\tilde\phi^{(i)}_{+0} &= \tilde\phi^{(i)}_{+0/2} + \frac{\delta\tilde\eta}{2}\,\tilde\pi^{(i)}_{+0}a^{-(3-\alpha)}_{+0}\,.\\
HC \quad &: \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left(\widetilde E_K + \widetilde E_G + \widetilde E_V\right),
\end{aligned}
$$

$$\tilde\pi_i' = a^{1+\alpha}\tilde\nabla^2\tilde\phi_i - a^{3+\alpha}\tilde V_{,\tilde\phi_i} \qquad\qquad \tilde\phi_i' = a^{-(3-\alpha)}\tilde\pi_i\,,$$

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$$\left\{\begin{aligned} \tilde\pi^{(i)}_{+0} &= \tilde\pi^{(i)} + \left(a^{1+\alpha}_{+0/2}\sum_k \widetilde\Delta^-_k\widetilde\Delta^+_k\tilde\phi^{(i)}_{+0/2} - a^{3+\alpha}_{+0/2}\widetilde V_{,\tilde\phi^{(i)}}\Big|_{+0/2}\right)\delta\tilde\eta\,,\\ b_{+0} &= b + \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\Big[(\alpha-2)\overline{\widetilde E_K} + \alpha\widetilde E_G + (\alpha+1)\widetilde E_V\Big]\delta\tilde\eta\,, \end{aligned}\right.$$

$$\left\{\begin{aligned} a_{+0} &= a_{+0/2} + b_{+0}\,\frac{\delta\tilde\eta}{2}\,,\\ \tilde\phi^{(i)}_{+0} &= \tilde\phi^{(i)}_{+0/2} + \frac{\delta\tilde\eta}{2}\,\tilde\pi^{(i)}_{+0}a_{+0}^{-(3-\alpha)}\,. \end{aligned}\right.$$

$$HC \quad : \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left(\widetilde E_K + \widetilde E_G + \widetilde E_V\right),$$

$$\tilde\pi_i' = a^{1+\alpha}\,\tilde\nabla^2\tilde\phi_i - a^{3+\alpha}\tilde V_{,\tilde\phi_i} \qquad\qquad \tilde\phi_i' = a^{-(3-\alpha)}\tilde\pi_i\,,$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Integration

**"The Art" (2006.15122) presented 2 Verlet methods:**

| Label | Discussed here | Order | CosmoLattice |
|---|---|---|---|
| I. Velocity- | ✓ | $\mathcal{O}(dt^2)$ | ✓ |
| II. Position- | ✓ | $\mathcal{O}(dt^2)$ | ✓ |

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Verlet Integration

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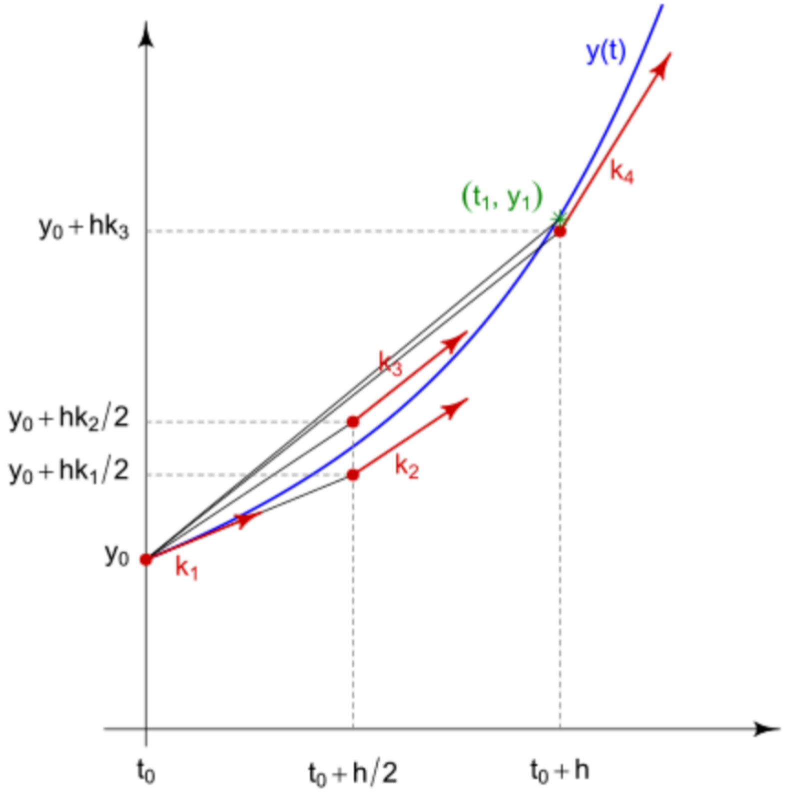
| I. Velocity- | ✓ | $\mathcal{O}(dt^2)$ | ✓ |
| II. Position- | ✓ | $\mathcal{O}(dt^2)$ | ✓ |

(Slightly faster)

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta




# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

### (Can solve non-symplectic systems)

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

## (Can solve non-symplectic systems)

**Continuum Problem:**

$$\begin{cases} a' & = & b\,, \\ \tilde{\phi}_i' & = & \tilde{\pi}_i\,, \\ \tilde{\pi}_i' & = & \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i]\,, \quad \text{(fld Kernels)} \\ b' & = & \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V]\,, \quad \text{(expansion Kernels)} \end{cases}$$

$$i = 1,2,....,N_s$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

## (Can solve non-symplectic systems)

**Continuum Problem:**

non-conservative !

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}'_i &= \tilde{\pi}_i\,, \\ \tilde{\pi}'_i &= \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i]\,, \quad \text{(fld Kernels)} \\ b' &= \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V]\,, \quad \text{(expansion Kernels)} \end{cases}$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

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$i = 1,2,\ldots,N_s$

where

$$\begin{cases} \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i] \equiv a^{-2(1-\alpha)}\widetilde{\nabla}^2\tilde{\phi}_i - (3-\alpha)\frac{b}{a}\tilde{\pi}_i - a^{2\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}}\sum_i\left\langle\tilde{\pi}_i^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

## (Can solve non-symplectic systems)

## Many flavours: $\mathcal{O}(d\eta^p)$ + explicit OR implicit*

**Continuum Problem:**

non-conservative !

$$\begin{cases} a' = b \,, \\ \tilde{\phi}_i' = \tilde{\pi}_i \,, \\ \tilde{\pi}_i' = \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i] \,, & \text{(fld Kernels)} \\ b' = \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V] \,, & \text{(expansion Kernels)} \end{cases} \qquad i = 1,2,....,N_s$$

where
$$\begin{cases} \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i] \equiv a^{-2(1-\alpha)} \widetilde{\nabla}^2 \tilde{\phi}_i - (3-\alpha)\frac{b}{a}\tilde{\pi}_i - a^{2\alpha}\tilde{V}_{,\tilde{\phi}_i} \,, \\ \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}}\sum_i \langle \tilde{\pi}_i^2 \rangle \,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle (\widetilde{\nabla}_k \tilde{\phi}_i)^2 \right\rangle \,, \quad \widetilde{E}_V \equiv \left\langle \widetilde{V}(\{\tilde{\phi}_j\}) \right\rangle \end{cases}$$

*Gauss-Legendre

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

### (Can solve non-symplectic systems)

## Explicit RK2 [ $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left.\begin{aligned}
\tilde{\pi}_i^{(p)} &\equiv \tilde{\pi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} k_i^{(p-1)}\,,\\
b^{(p)} &\equiv b(n_0) + \delta\tilde{\eta} c_{p,p-1} k_a^{(p-1)}\,,\\
\tilde{\phi}_i^{(p)} &\equiv \tilde{\phi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} \tilde{\pi}_i^{(p-1)}\,,\\
a^{(p)} &\equiv a(n_0) + \delta\tilde{\eta} c_{p,p-1} b^{(p-1)}\,,\\
k_i^{(p)} &\equiv \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}, \tilde{\pi}_i^{(p)}]\,,\\
k_a^{(p)} &\equiv \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\,,
\end{aligned}\right\}_{p=1,2} \Longrightarrow \left\{\begin{aligned}
\widetilde{\Delta}_0^+ \tilde{\phi}_i(\mathbf{n}, n_0) &= \tfrac{1}{2}(\tilde{\pi}_i^{(1)} + \tilde{\pi}_i^{(2)})\\
\Delta_0^+ a(n_0) &= \tfrac{1}{2}(b^{(1)} + b^{(2)})\\
\widetilde{\Delta}_0^+ \tilde{\pi}_i(\mathbf{n}, n_0) &= \tfrac{1}{2}(k_i^{(1)} + k_i^{(2)})\\
\widetilde{\Delta}_0^+ b(n_0) &= \tfrac{1}{2}(k_a^{(1)} + k_a^{(2)})
\end{aligned}\right.$$

$$\left(\; c_{10} \equiv 0\,, \qquad c_{21} \equiv 1 \;\right)$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Runge-Kutta

### (Can solve non-symplectic systems)

## Explicit RK4 [ $\mathcal{O}(d\eta^4)$ ]

**Lattice Problem:**

$$\left.\begin{array}{rcl}
\tilde{\pi}_i^{(p)} & \equiv & \tilde{\pi}_i(\mathbf{n},n_0) + \delta\tilde{\eta} c_{p,p-1} k_i^{(p-1)}\,, \\
b^{(p)} & \equiv & b(n_0) + \delta\tilde{\eta} c_{p,p-1} k_a^{(p-1)}\,, \\
\tilde{\phi}_i^{(p)} & \equiv & \tilde{\phi}_i(\mathbf{n},n_0) + \delta\tilde{\eta} c_{p,p-1} \tilde{\pi}_i^{(p-1)}\,, \\
a^{(p)} & \equiv & a(n_0) + \delta\tilde{\eta} c_{p,p-1} b^{(p-1)}\,, \\
k_i^{(p)} & \equiv & \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}, \tilde{\pi}_i^{(p)}]\,, \\
k_a^{(p)} & \equiv & \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\,,
\end{array}\right\}_{p=1,2,3,4}
\Longrightarrow
\left\{\begin{array}{rcl}
\widetilde{\Delta}_0^+ \tilde{\phi}_i(\mathbf{n},n_0) & = & \frac{1}{6}(\tilde{\pi}_i^{(1)} + 2\tilde{\pi}_i^{(2)} + 2\tilde{\pi}_i^{(3)} + \tilde{\pi}_i^{(4)})\,, \\
\Delta_0^+ a(n_0) & = & \frac{1}{6}(b^{(1)} + 2b^{(2)} + 2b^{(3)} + b^{(4)}) \\
\widetilde{\Delta}_0^+ \tilde{\pi}_i(\mathbf{n},n_0) & = & \frac{1}{6}(k_i^{(1)} + 2k_i^{(2)} + 2k_i^{(3)} + k_i^{(4)}) \\
\widetilde{\Delta}_0^+ b(n_0) & = & \frac{1}{6}(k_a^{(1)} + 2k_a^{(2)} + 2k_a^{(3)} + k_a^{(4)})
\end{array}\right.$$

$$\left( c_{10} \equiv 0\,, \quad c_{21} \equiv \frac{1}{2}\,, \quad c_{32} \equiv \frac{1}{2}\,, \quad c_{43} \equiv 1 \right)$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators (Yoshida Method)

$$[\ \mathcal{O}(d\eta^4), \mathcal{O}(d\eta^6), \mathcal{O}(d\eta^8), \mathcal{O}(d\eta^{10}), \ \ldots\ ]$$

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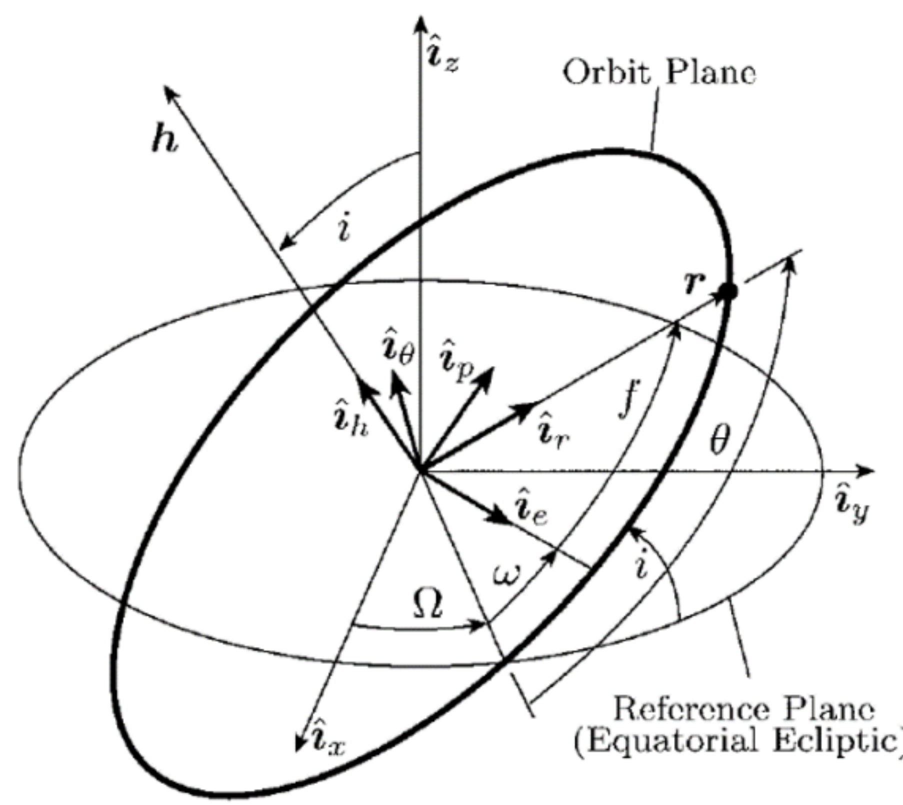

# Scalar Field @ Expanding Background (SF@EB) Algorithms

# Higher Order Symplectic Integrators

**Continuum Problem:**

conservative ! (good for symplectic integrators)

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' &= \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

**Continuum Problem:**

conservative ! (good for symplectic integrators)

$$\begin{cases} a' &=& b\,, \\ \tilde{\phi}_i' &=& a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' &=& \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' &=& \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{cases}$$

where

$$\begin{cases} \mathcal{K}_i[a,\{\tilde{\phi}_j\}] \equiv a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^6}\sum_i\langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle\,. \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Continuum Problem:**

conservative ! (good for symplectic integrators)

$$\left\{\begin{array}{rcl} a' &=& b\,, \\ \tilde{\phi}_i' &=& a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' &=& \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' &=& \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{array}\right.$$

where

$$\left\{\begin{array}{l} \mathcal{K}_i[a,\{\tilde{\phi}_j\}] \equiv a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V] \equiv \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i\langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle\,. \end{array}\right.$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

**[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]**

**Continuum Problem:**

**conservative !** (good for symplectic integrators)

$$\begin{cases} a' = b\,, \\ \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' = \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{cases}$$

where

$$\begin{cases} \mathcal{K}_i[a,\{\tilde{\phi}_j\}] \equiv a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^6}\sum_i\langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle\,. \end{cases}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

**[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]**

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

# Higher Order Symplectic Integrators

## [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

1/2 - step

$$\left\{\begin{array}{l}\tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0)\\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0)\\ a^{(0)} \equiv a(n_0)\\ b^{(0)} \equiv b(n_0)\end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b^{(p)}_{1/2} &=& b^{(p-1)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}]\\ \tilde{\pi}^{(p)}_{i,1/2} &=& \tilde{\pi}_i^{(p-1)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}]\\ a^{(p)}_{1/2} &=& a^{(p-1)} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde{\eta}}{2}\\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p\delta\tilde{\eta}\,\tilde{\pi}^{(p)}_{i,1/2}(a^{(p)}_{1/2})^{-(3-\alpha)}\,,\\ a^{(p)} &=& a^{(p)}_{1/2} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde{\eta}}{2}\,,\\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}^{(p)}_{i,1/2} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}]\\ b^{(p)} &=& b^{(p)}_{1/2} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

**Full - step**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

**[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]**

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow$$

**1/2 - step**

$$\left\{\begin{array}{rcl} b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

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# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b^{(p)}_{1/2} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}^{(p)}_{i,1/2} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a^{(p)}_{1/2} &=& a^{(p-1)} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}^{(p)}_{i,1/2}(a^{(p)}_{1/2})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a^{(p)}_{1/2} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}^{(p)}_{i,1/2} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b^{(p)}_{1/2} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

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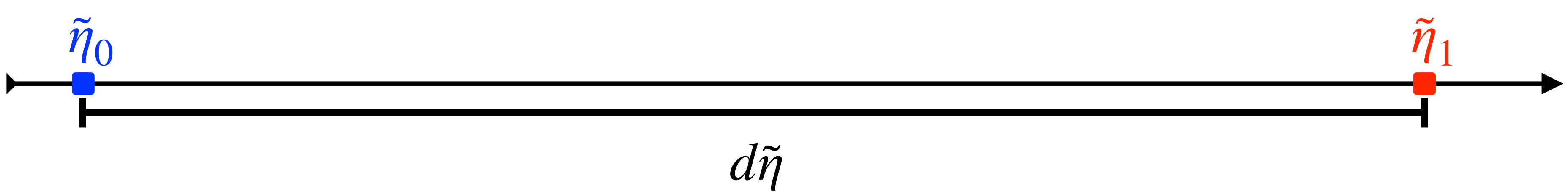

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)} , \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2} , \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

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$\tilde{\eta}_0$ $\tilde{\eta}_1$

$$d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$
\left\{\begin{array}{l}
\tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\
\tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\
a^{(0)} \equiv a(n_0) \\
b^{(0)} \equiv b(n_0)
\end{array}\right.
\Longrightarrow
\left\{\begin{array}{rcl}
b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\
\tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\
a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2} \\
\tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\
a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2}\,, \\
\tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\
b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]
\end{array}\right\}_{p=1,\ldots,s}
\Longrightarrow
$$

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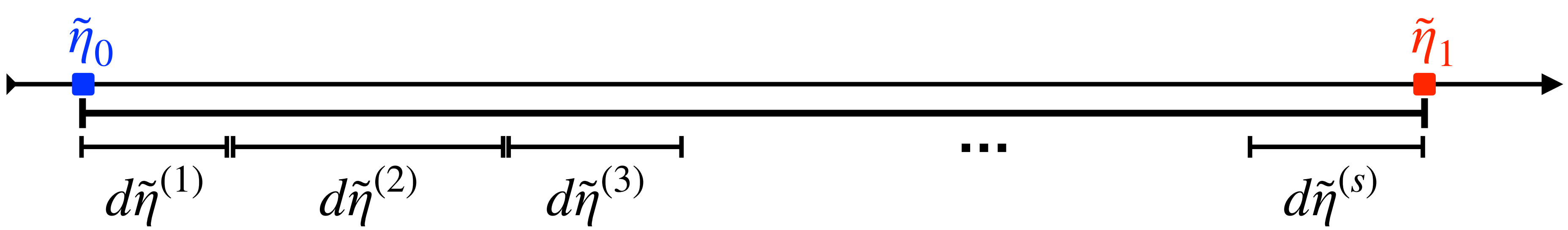


$$
d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)} \quad ; \quad
\left\{\begin{array}{l}
d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\
\sum_{p=1}^{s} \omega_p = 1 \qquad (\,\omega_p < 0 \text{ allowed})
\end{array}\right.
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p\delta\tilde{\eta}\,\tilde{\pi}_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

**(s-copies)**

$$\tilde{\eta}_0 \qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)} \qquad \tilde{\eta}_1$$

$$\left\{\begin{array}{l} d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\ \sum_{p=1}^{s} \omega_p = 1 \end{array}\right.$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

**(s-copies)**

$$\tilde{\eta}_0 \qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)} \qquad \tilde{\eta}_1$$

$$\Longrightarrow \left\{\begin{array}{l} \tilde{\pi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)}\,. \end{array}\right.$$

$$\left\{\begin{array}{l} d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\ \sum_{p=1}^{s} \omega_p = 1 \end{array}\right.$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

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**(s-copies)**

$$\tilde{\eta}_0 \xrightarrow{\;d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}\;} \tilde{\eta}_1$$

**Appropriate $\{\omega_p\}_{p=1,\dots,s}$ : Errors intermediate steps cancel to $\mathcal{O}(\delta\eta^r)$**

$$s_n \equiv 2s_{n-1} + 1 = 1,3,7,15,\ldots \qquad \Longrightarrow \qquad \mathcal{O}(\delta\eta^{2n})$$

$$(s_1 \equiv 1\ ;\ n = 1,\ 2,\ \ldots) \qquad\qquad (r = 2,4,6,8,\ldots)$$

$$\left\{\begin{array}{l} d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\ \sum_{p=1}^{s}\omega_p = 1 \end{array}\right.$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

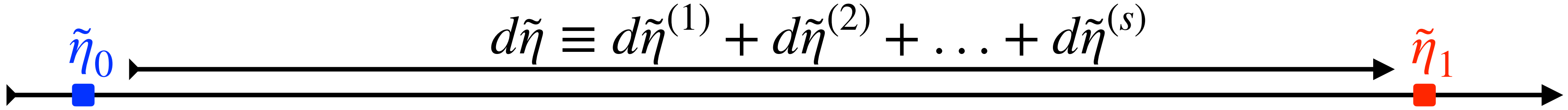
## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\boxed{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p\delta\tilde{\eta}\,\tilde{\pi}_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

**(s-copies)**




**Appropriate** $\{\omega_p\}_{p=1,\dots,s}$ **: Errors intermediate steps cancel to** $\mathcal{O}(\delta\eta^r)$

$$\left\{\begin{array}{l} d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\ \sum_{p=1}^{s}\omega_p = 1 \end{array}\right.$$

**e.g.** $\begin{array}{c} w_1 = w_3 = 1.3512071919596577718181 \\ w_2 = -1.7024143838758382002464 \end{array}$ $\quad \mathcal{O}(d\eta^4)$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\begin{cases} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{cases} \quad \underset{\text{(s-copies VV2)}}{\Longrightarrow} \quad \begin{cases} \tilde{\pi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)}\,. \end{cases}$$

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

$$\left[\ d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1\ \right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \tilde{\pi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)}\,. \end{array}\right.$$

**(s-copies VV2)**

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

$$\left[\; d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1 \;\right]$$

| Name | Order | $w_i = \frac{\delta t_i}{\delta t}$ | s |
|---|---|---|---|
| $VV4$ | $O(\delta t^4)$ | $w_1 = w_3 = 1.3512071919596577171818$<br>$w_2 = -1.7024143838758382002640$ | 3 |
| $VV6$ | $O(\delta t^6)$ | $w_1 = w_7 = 0.78451361047755726382$<br>$w_2 = w_6 = 0.23557321335935813368$<br>$w_3 = w_5 = -1.17767998417887100690$<br>$w_4 = 1.31518632068391121890$ | 7 |
| $VV8$ | $O(\delta t^8)$ | $w_1 = w_{15} = 0.74167036435061295345$<br>$w_2 = w_{14} = -0.40910082580003159400$<br>$w_3 = w_{13} = 0.19075471029623837995$<br>$w_4 = w_{12} = -0.57386247111608226666$<br>$w_5 = w_{11} = 0.29906418130365592384$<br>$w_6 = w_{10} = 0.33462491824529818378$<br>$w_7 = w_9 = 0.31529309239676659663$<br>$w_8 = -0.79688793935291635402$ | 15 |
| $VV10$ | $O(\delta t^{10})$ | $w_1 = w_{31} = -0.48159895600253002870$<br>$w_2 = w_{30} = 0.00363039315445959268790$<br>$w_3 = w_{29} = 0.50180317558723140279$<br>$w_4 = w_{28} = 0.28298402624506254868$<br>$w_5 = w_{27} = 0.80702967895372223806$<br>$w_6 = w_{26} = -0.026090580538592205447$<br>$w_7 = w_{25} = -0.87286590146318071547$<br>$w_8 = w_{24} = -0.52373568062510581643$<br>$w_9 = w_{23} = 0.44521844299952789252$<br>$w_{10} = w_{22} = 0.18612289547097907887$<br>$w_{11} = w_{21} = 0.23137327866438360633$<br>$w_{12} = w_{20} = -0.52191036590418628905$<br>$w_{13} = w_{19} = 0.74866113714499296793$<br>$w_{14} = w_{18} = 0.066736511890604057532$<br>$w_{15} = w_{17} = -0.80360324375670830316$<br>$w_{16} = 0.91249037635867994571$ | 31 |

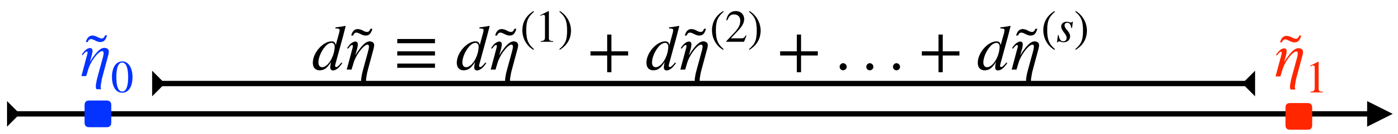
# Scalar Field @ Expanding Background (SF@EB) Algorithms

## Higher Order Symplectic Integrators

### [based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$ ]

**Lattice Problem:**

$$\left\{\begin{aligned} \tilde{\pi}_i^{(0)} &\equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} &\equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} &\equiv a(n_0) \\ b^{(0)} &\equiv b(n_0) \end{aligned}\right. \quad \underset{\text{(s-copies VV2)}}{\Longrightarrow} \quad \left\{\begin{aligned} \tilde{\pi}_i(\mathbf{n}, n_0+1) &\equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) &\equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) &\equiv a^{(s)} \\ b(n_0+1) &\equiv b^{(s)}\,. \end{aligned}\right.$$




$$\left[\; d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1 \;\right]$$

| Name | Order | $w_i = \frac{\delta t_i}{\delta t}$ | s |
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| $VV4$ | $O(\delta t^4)$ | $w_1 = w_3 = 1.351207191959657771818$<br>$w_2 = -1.702414403875838200264$ | 3 |
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# Summary

**LeapFrog (LF)**
**Verlet methods (VV2,PV2)**
**Runge-Kutta methods (RK2, RK4)**
**Higher Order integrators (VV4, VV6, VV8, VV10)**

# *CosmoLattice* – School 2022

# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

* **L4.a:** $\mathcal{O}(dt^2) - \mathcal{O}(dt^{10})$ **SF@EB Algorithms** ✔
* **L4.b: Models** $\tanh^p(\phi/M)$

# — Lecture 4 —

## Lattice Formulation of scalar field dynamics in an expanding background

* L4.a: $\mathcal{O}(dt^2) - \mathcal{O}(dt^{10})$ SF@EB Algorithms ✓
* L4.b: Models $\tanh^p(\phi/M)$

# — Lecture 4.b —

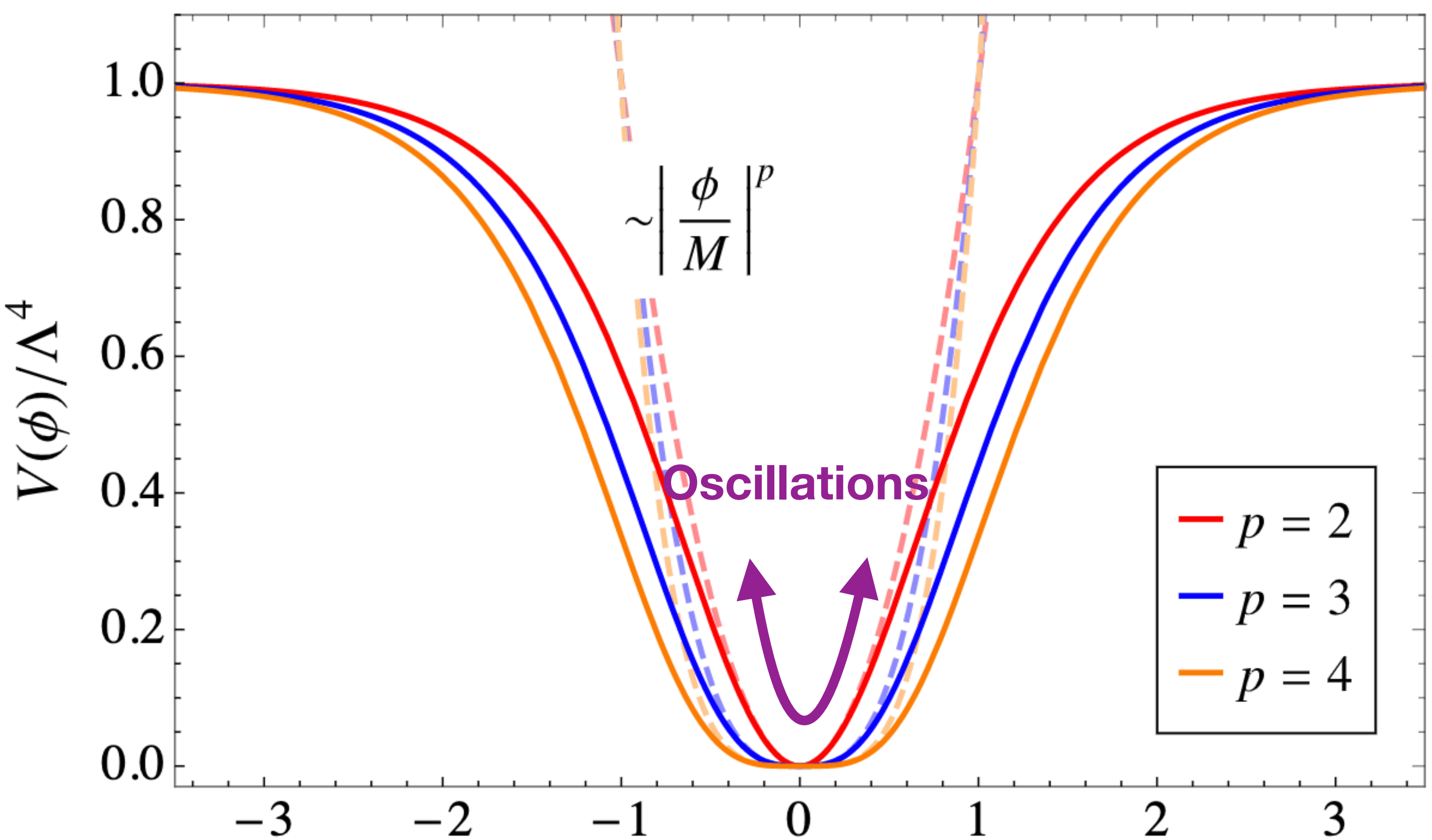
## Models

$$\tanh^p(\phi/M)$$

# **Inflationary Models** $\tanh^p(\phi/M)$

$$V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2 \qquad (p \geq 2)$$

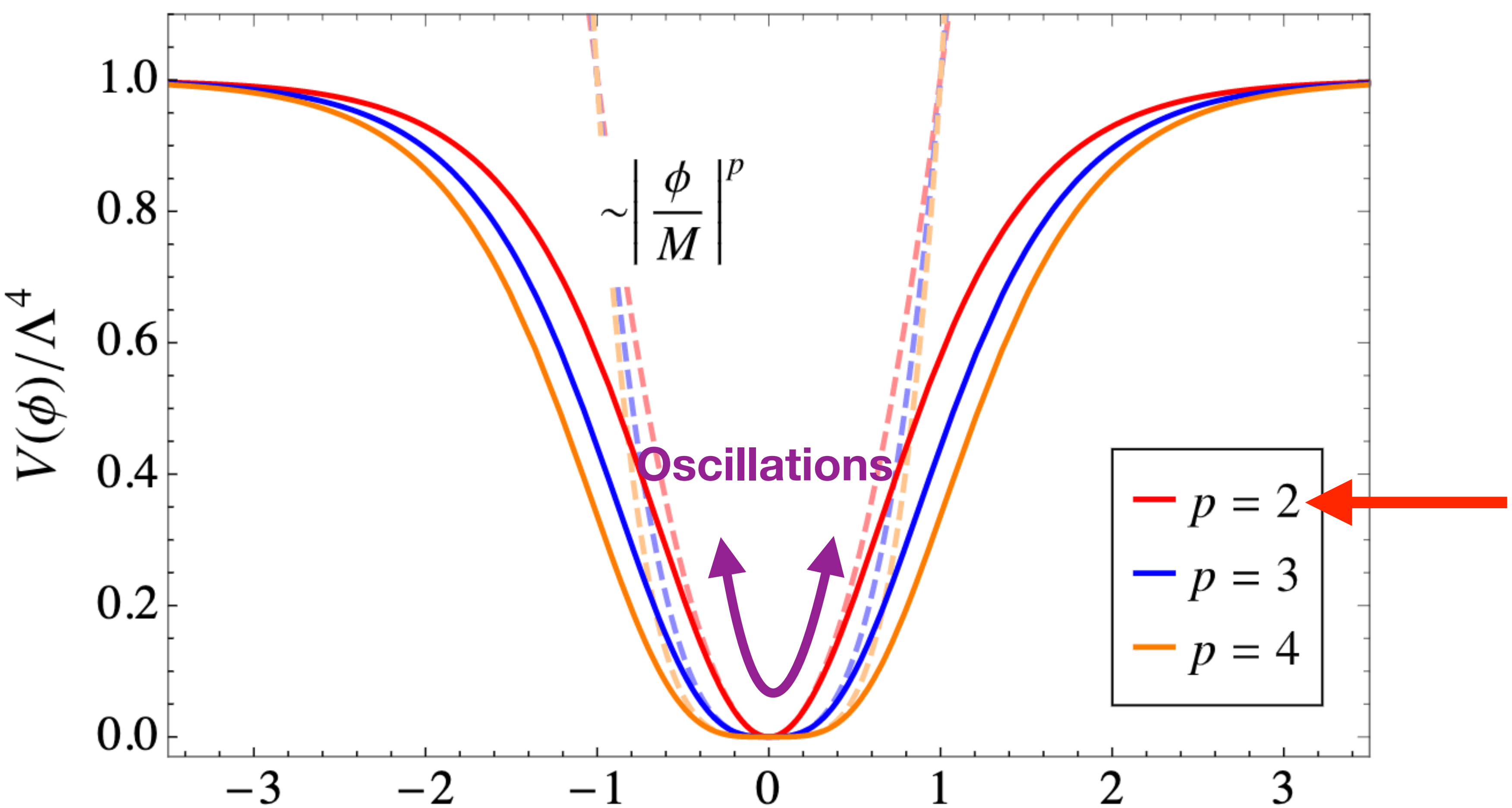
( *e.g.* $\alpha-$attractors, Kallosh, Linde 2013)



# Inflationary Models $\tanh^p(\phi/M)$

$$V(\phi,\chi) = \frac{1}{2}\Lambda^4 \left[\tanh\left(\frac{\phi}{M}\right)\right]^2 + \frac{1}{2}g^2\chi^2\phi^2$$

( *e.g.* $\alpha-$attractors, Kallosh, Linde 2013)



# Inflationary Models $\tanh^p(\phi/M)$

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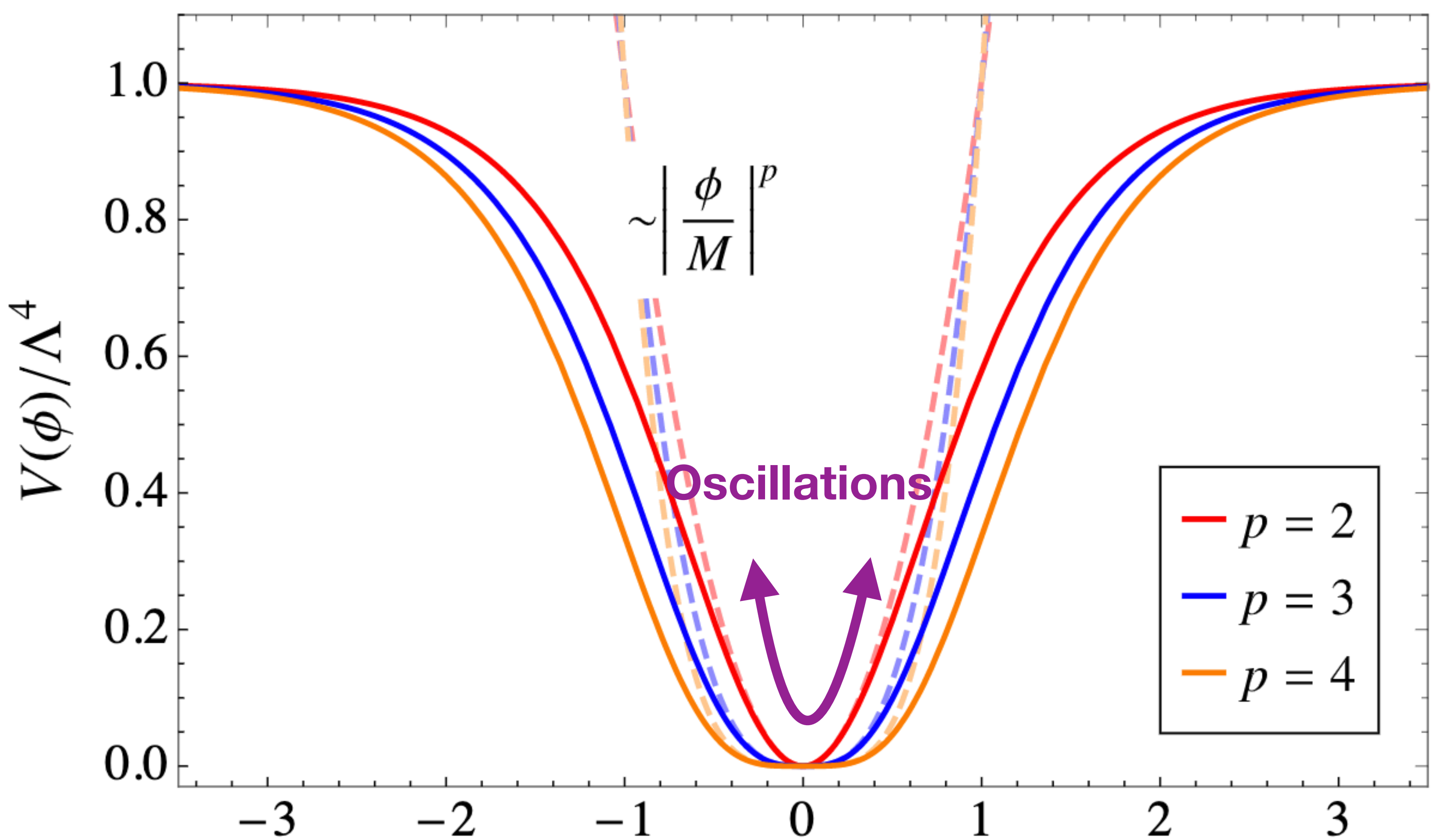

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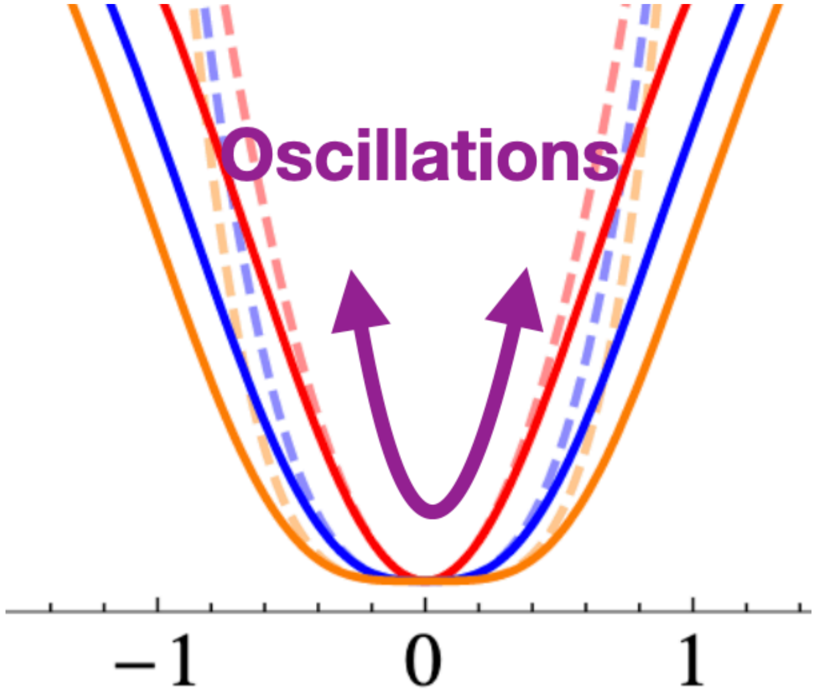
$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p\ , \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$



# Inflationary Models $\tanh^p(\phi/M)$

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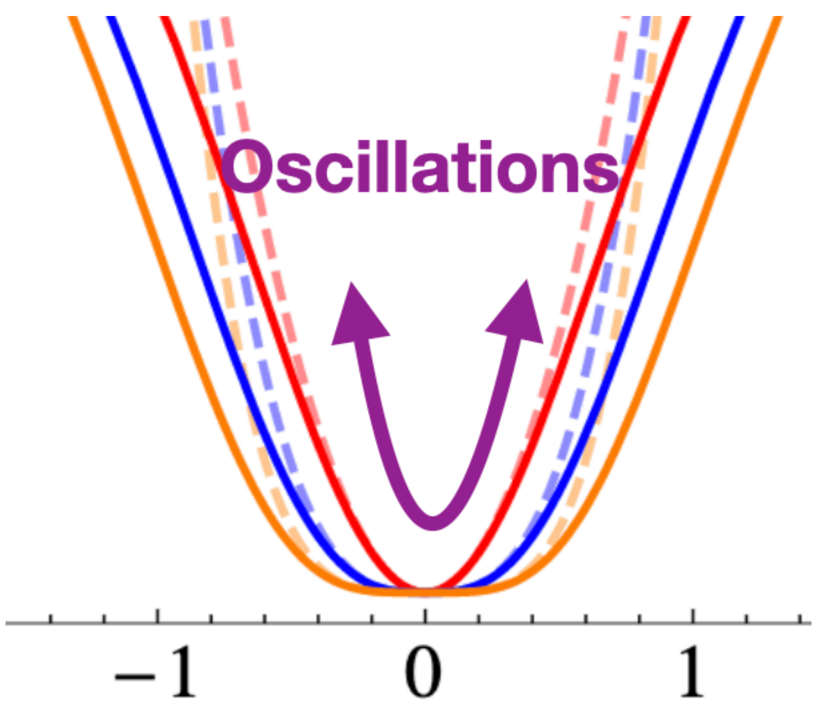


$$V_\phi \simeq \frac{1}{2}m_\phi^2\phi^2\ ;\quad m_\phi^2 \equiv \left(\frac{\Lambda}{M}\right)^2\Lambda^2$$

$$V_\phi \simeq \frac{1}{4}\lambda\phi^4\ ;\quad \lambda \equiv \left(\frac{\Lambda}{M}\right)^4$$

$$V_\phi \simeq \frac{1}{6}\frac{1}{\sigma^2}\phi^6\ ;\quad \sigma^2 \equiv \left(\frac{M}{\Lambda}\right)^4 M^2$$

# Inflationary Models $\tanh^p(\phi/M)$

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4 \left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \quad (p \geq 2)$$

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**Homogeneous:** $\ddot{\phi} + 3H\dot{\phi} + \Omega^2(|\phi|)\phi = 0\ ,\quad \boxed{\Omega^2(|\phi|) \equiv \lambda\mu^{4-p}|\phi|^{p-2}\ ,}$

# Inflationary Models $\tanh^p(\phi/M)$

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$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p\ , \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$

**Homogeneous:** $\ddot\phi + 3H\dot\phi + \Omega^2(|\phi|)\phi = 0\ ,\quad \boxed{\Omega^2(|\phi|) \equiv \lambda\mu^{4-p}|\phi|^{p-2}\ ,}$

Turner '83: $\Omega(\phi) \sim \omega_*\left(\frac{t}{t_*}\right)^{4/p-2} \sim \left(\frac{a}{a_*}\right)^{\frac{-3(p-2)}{(p+2)}}\ ,\qquad \omega_* \equiv \sqrt{\lambda}\mu^{\frac{4-p}{2}}\phi_*^{\frac{p-2}{2}}$

**Initial amplitude**
**(@ oscillations onset)**

# Inflationary Models $\tanh^p(\phi/M)$

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \qquad (p \geq 2)$$

( e.g. $\alpha$ – attractors, Kallosh, Linde 2013)

$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p\,, \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$

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Turner '83: $\Omega(\phi) \sim \omega_*\left(\frac{t}{t_*}\right)^{4/p-2} \sim \left(\frac{a}{a_*}\right)^{\frac{-3(p-2)}{(p+2)}}\,, \qquad \omega_* \equiv \sqrt{\lambda}\mu^{\frac{4-p}{2}}\phi_*^{\frac{p-2}{2}}$

**Choice Param.:** $\boxed{\alpha = 3\left(\frac{p-2}{p+2}\right)\,, \quad f_* \equiv \phi_*\,, \quad \omega_* \equiv \Lambda^2 M^{-\frac{p}{2}}\phi_*^{\frac{p-2}{2}}}$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$
$$\tilde{\phi} = \frac{\phi}{f_*}$$

# Inflationary Models $\tanh^p(\phi/M)$

$$V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2 \qquad (p \geq 2)$$

( *e.g.* $\alpha$ – attractors, Kallosh, Linde 2013)

## Exercise time !

**Run** for p = 2, p = 4, p = 6,

**changing** $k_{\rm IR}, k_{\rm UV}, N,$

**choosing** LF, VV2, VV4, VV6, VV8, VV10

**checking** energy conservation, field mean values, field variance, field spectra, …

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{|\phi|}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \qquad (p \geq 2)$$

$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{|\phi|}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p\,, \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$

**Choice Param.:** $\boxed{\alpha = 3\left(\frac{p-2}{p+2}\right)\,, \quad f_* \equiv \phi_*\,, \quad \omega_* \equiv \Lambda^2 M^{-\frac{p}{2}}\phi_*^{\frac{p-2}{2}}}$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$
$$\tilde{\phi} = \frac{\phi}{f_*}\;;\; \tilde{\chi} = \frac{\chi}{f_*}$$

**Potential:** $$\tilde{V}(\tilde{\phi},\tilde{\chi}) \equiv \frac{1}{f_*^2\omega_*^2}V(\tilde{\phi},\tilde{\chi}) = \frac{1}{p}\tilde{M}^p\tanh^p\left(\frac{|\tilde{\phi}|}{\tilde{M}}\right) + \frac{1}{2}q\,\tilde{\chi}^2\tilde{\phi}^2\;; \quad \begin{cases} \tilde{M} \equiv \frac{M}{\phi_*}\;;\; \tilde{\Lambda} \equiv \frac{\Lambda}{\phi_*} \\ q \equiv \frac{g^2\phi_*^2}{\omega_*^2} = \frac{g^2\tilde{M}^p}{\tilde{\Lambda}^4} \end{cases}$$

**Deriv. Potential:** $$\frac{\partial\tilde{V}}{\partial\tilde{\phi}} = \tilde{M}^{p-1}\frac{\tanh^{p-1}(|\tilde{\phi}|/\tilde{M})}{\cosh^2(|\tilde{\phi}|/M)}\mathrm{sgn}(\tilde{\phi}) + q\,\tilde{\chi}^2\tilde{\phi}\;; \quad \frac{\partial\tilde{V}}{\partial\tilde{\chi}} = q\,\tilde{\phi}^2\tilde{\chi}$$

**Deriv.^2 Potential:** $$\begin{cases} \frac{\partial^2\tilde{V}}{\partial\tilde{\phi}^2} = \frac{\tilde{M}^{p-2}}{\cosh^2(|\tilde{\phi}|/M)}\left[(p-1)\tanh^{p-2}(|\tilde{\phi}|/\tilde{M}) - (p+1)\tanh^p(|\tilde{\phi}|/\tilde{M})\right]\mathrm{sgn}(\tilde{\phi}) + q\,\tilde{\chi}^2 \\ \frac{\partial^2\tilde{V}}{\partial\tilde{\chi}^2} = q\,\tilde{\phi}^2 \end{cases}$$

# Back slides

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$
\begin{aligned}
IC \quad &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} \quad &= \quad a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} \quad &= \quad \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} + a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}_k^- \widetilde{\Delta}_k^+ \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta}
\end{aligned}
$$

---

$$
\widetilde{\Delta}_0^- [a^{3-\alpha}_{+0/2} \tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}_i^- \widetilde{\Delta}_i^+ \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b} ; \qquad b = 1,2,\ldots,N_s
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}^+_0 \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$
\begin{aligned}
IC &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} &= a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} &= \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} + a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}^-_k \widetilde{\Delta}^+_k \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta} \\
\tilde{\phi}^{(a)}_{+0} &= \tilde{\phi}^{(a)} + \delta\tilde{\eta}\, \tilde{\pi}^{(a)}_{+0/2}
\end{aligned}
$$

---

$$
\widetilde{\Delta}^-_0 [a^{3-\alpha}_{+0/2} \tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}^-_i \widetilde{\Delta}^+_i \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}^+_0 \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$\begin{aligned}
IC &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} &= a_{-0/2} + b\,\delta\tilde{\eta} \qquad \longrightarrow \qquad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} &= \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} \;+\; a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}^-_k \widetilde{\Delta}^+_k \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta} \\
\tilde{\phi}^{(a)}_{+0} &= \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2}
\end{aligned}$$

---

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}} \sum_b \left\langle (\widetilde{\Delta}^+_0 \tilde{\phi}_b)^2 \right\rangle , \quad \widetilde{E}_G \equiv \frac{1}{2a^2} \sum_{b,k} \left\langle (\widetilde{\Delta}^+_k \tilde{\phi}_b)^2 \right\rangle , \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad\right]$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

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$$
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IC \quad &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} \quad &= \quad a_{-0/2} + b\,\delta\tilde{\eta} \quad \longrightarrow \quad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} \quad &= \quad \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} \; + \; a^{-(3-\alpha)}_{+0/2}\left(a^{1+\alpha}\sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\
\tilde{\phi}^{(a)}_{+0} \quad &= \quad \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2} \\
b_{+0} \quad &= \quad b + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right],
\end{aligned}
$$

$\overline{\widetilde{E}_G} \rightarrow \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}$ $\qquad$ $\overline{\widetilde{E}_V} \rightarrow \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$

---

$$
\left[\; \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle(\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \;\right]
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

$$
\begin{aligned}
IC \quad &: \quad \{\tilde{\phi}_a, b\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, a_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta} \\
a_{+0/2} \quad &= \quad a_{-0/2} + b\,\delta\tilde{\eta} \qquad \longrightarrow \qquad a \equiv (a_{+0/2} + a_{-0/2})/2 \\
\tilde{\pi}^{(a)}_{+0/2} \quad &= \quad \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} \; + \; a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}_k^- \widetilde{\Delta}_k^+ \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta} \\
\tilde{\phi}^{(a)}_{+0} \quad &= \quad \tilde{\phi}^{(a)} + \delta\tilde{\eta}\, \tilde{\pi}^{(a)}_{+0/2} \\
b_{+0} \quad &= \quad b + \frac{\delta\tilde{\eta}}{3} \left(\frac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha} \left[ (\alpha - 2)\widetilde{E}_K + \alpha \overline{\widetilde{E}_G} + (\alpha + 1)\overline{\widetilde{E}_V} \right],
\end{aligned}
$$

$$\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

$$
HC \quad : \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)} \left( \overline{\widetilde{E}_K} + \widetilde{E}_G + \widetilde{E}_V \right).
$$

$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

---

$$
\left[ \quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}} \sum_b \left\langle (\widetilde{\Delta}_0^+ \tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2} \sum_{b,k} \left\langle (\widetilde{\Delta}_k^+ \tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad \right]
$$

# Scalar Field @ Expanding Background (SF@EB) Algorithms

## (Staggered) LeapFrog

*I) Iterative scheme for* $\tilde{\pi}^{(a)}_{+0/2} \equiv \widetilde{\Delta}_0^+ \tilde{\phi}_a$ and scale factor $a(n_0 + 1/2)$:

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\tilde{\pi}^{(a)}_{+0/2} \quad &= \quad \left(\frac{a_{-0/2}}{a_{+0/2}}\right)^{3-\alpha} \tilde{\pi}^{(a)}_{-0/2} \; + \; a_{+0/2}^{-(3-\alpha)} \left( a^{1+\alpha} \sum_k \widetilde{\Delta}_k^- \widetilde{\Delta}_k^+ \tilde{\phi}^{(a)} - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}^{(a)}} \right) \delta\tilde{\eta} \\
\tilde{\phi}^{(a)}_{+0} \quad &= \quad \tilde{\phi}^{(a)} + \delta\tilde{\eta}\, \tilde{\pi}^{(a)}_{+0/2} \\
b_{+0} \quad &= \quad b + \frac{\delta\tilde{\eta}}{3} \left(\frac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha} \left[ (\alpha - 2)\widetilde{E}_K + \alpha \overline{\widetilde{E}_G} + (\alpha + 1)\overline{\widetilde{E}_V} \right], \\
HC \quad &: \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)} \left( \overline{\widetilde{E}_K} + \widetilde{E}_G + \widetilde{E}_V \right).
\end{aligned}
$$

$$\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

---

$$\left[ \quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}} \sum_b \left\langle (\widetilde{\Delta}_0^+ \tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2} \sum_{b,k} \left\langle (\widetilde{\Delta}_k^+ \tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad \right]$$