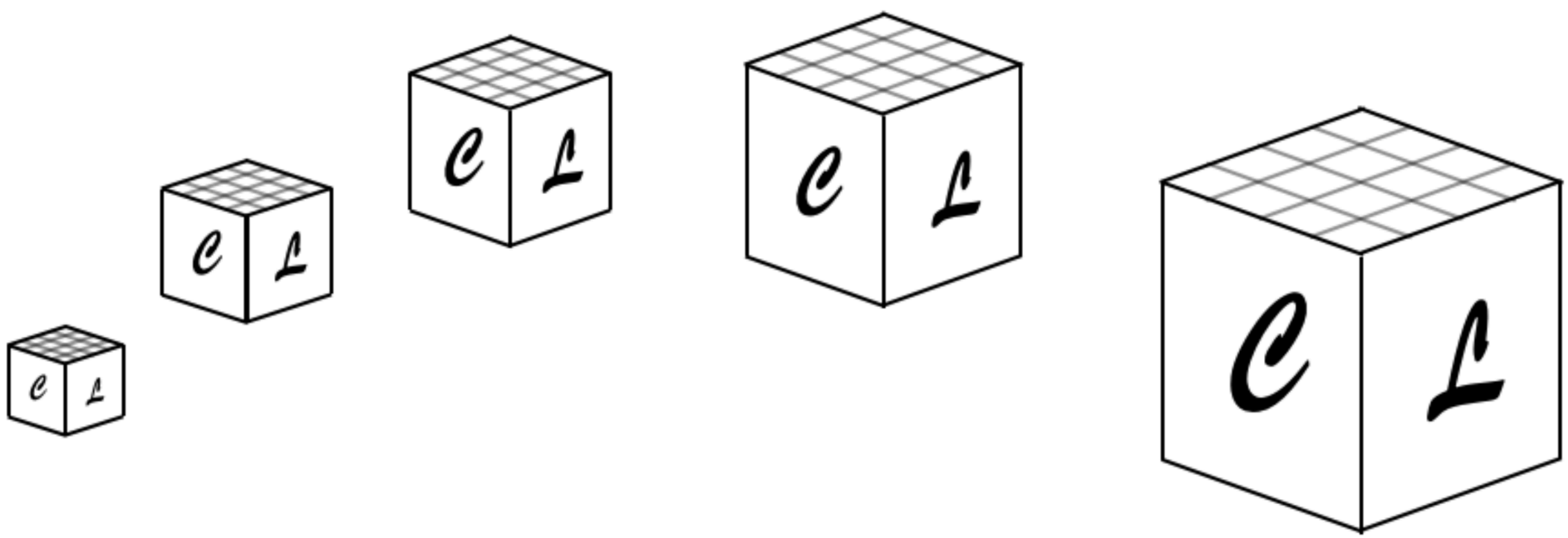
CosmoLattice

— School 2022 —



CosmoLattice

— School 2022 —

Bienvenidos !

Welcome !

Ben arrivato !

Bienvenue !

Willkommen !

ברוכים הבאים !

欢迎 !

Benvinguts!

ようこそ !

Welkom !

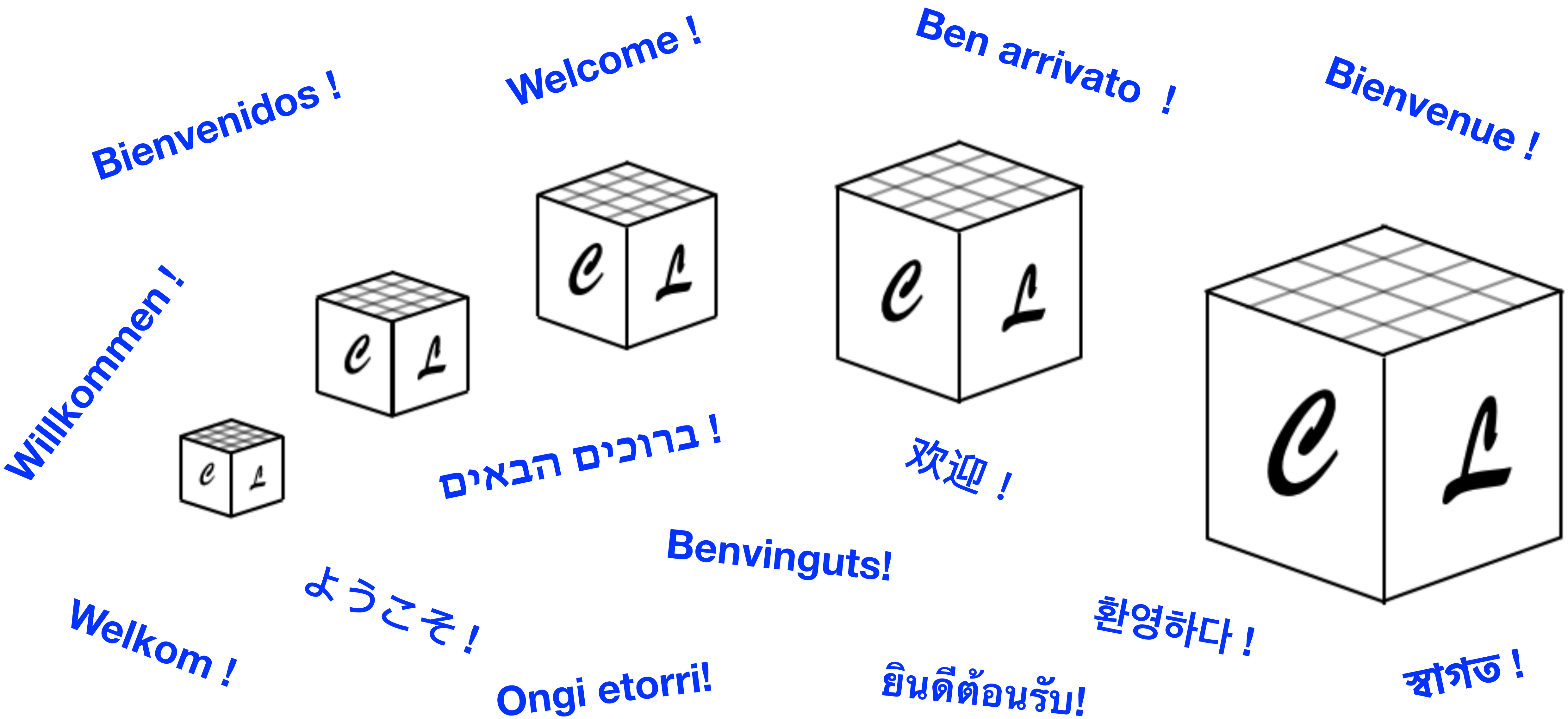
Ongi etorri!

ยินดีต้อนรับ!

환영하다 !

স্বাগত !

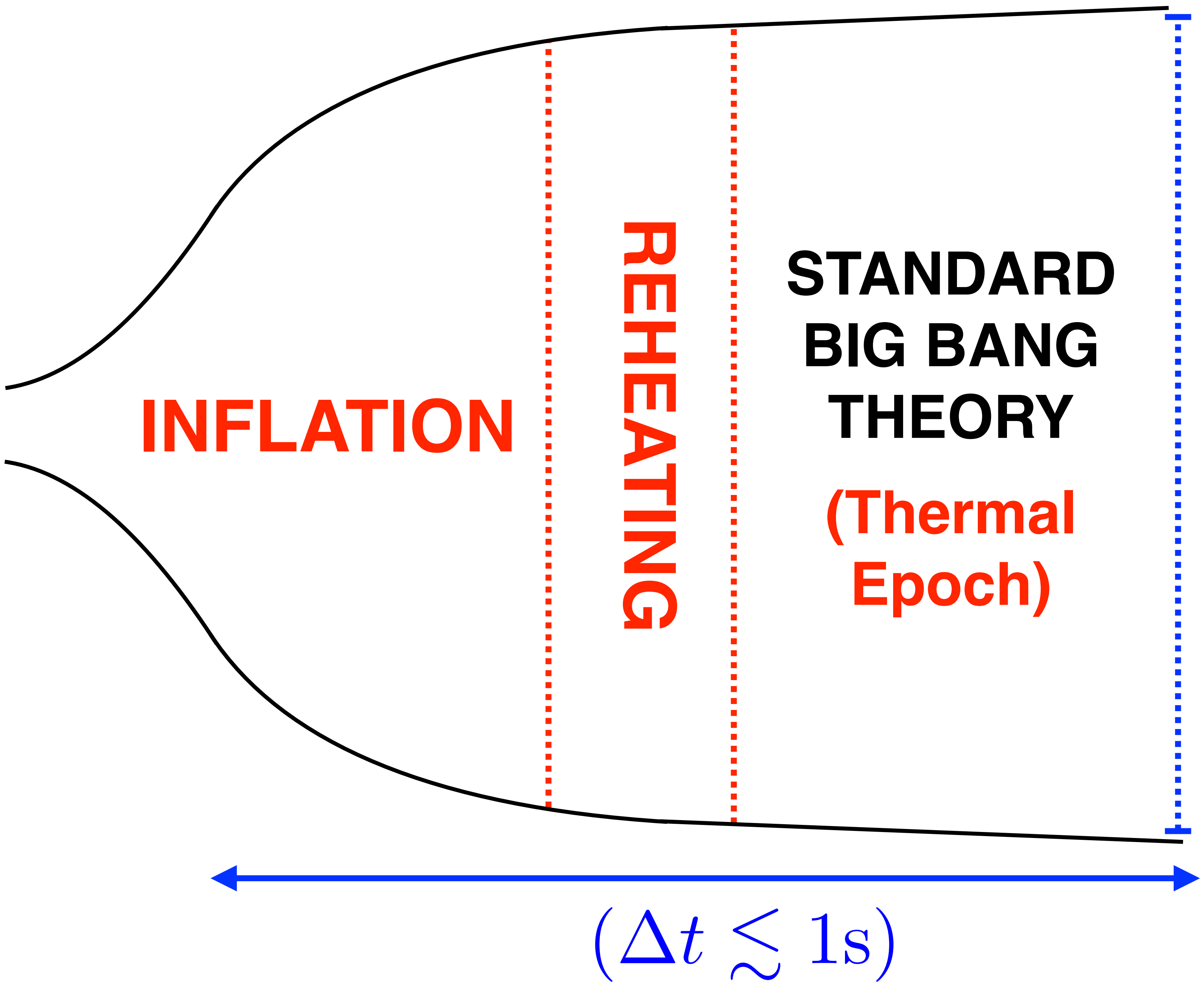
The Art of Simulating the Early Universe



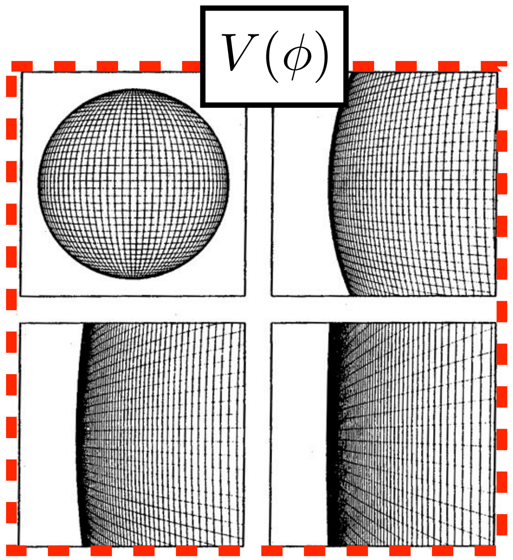
The Art of Simulating the Early Universe

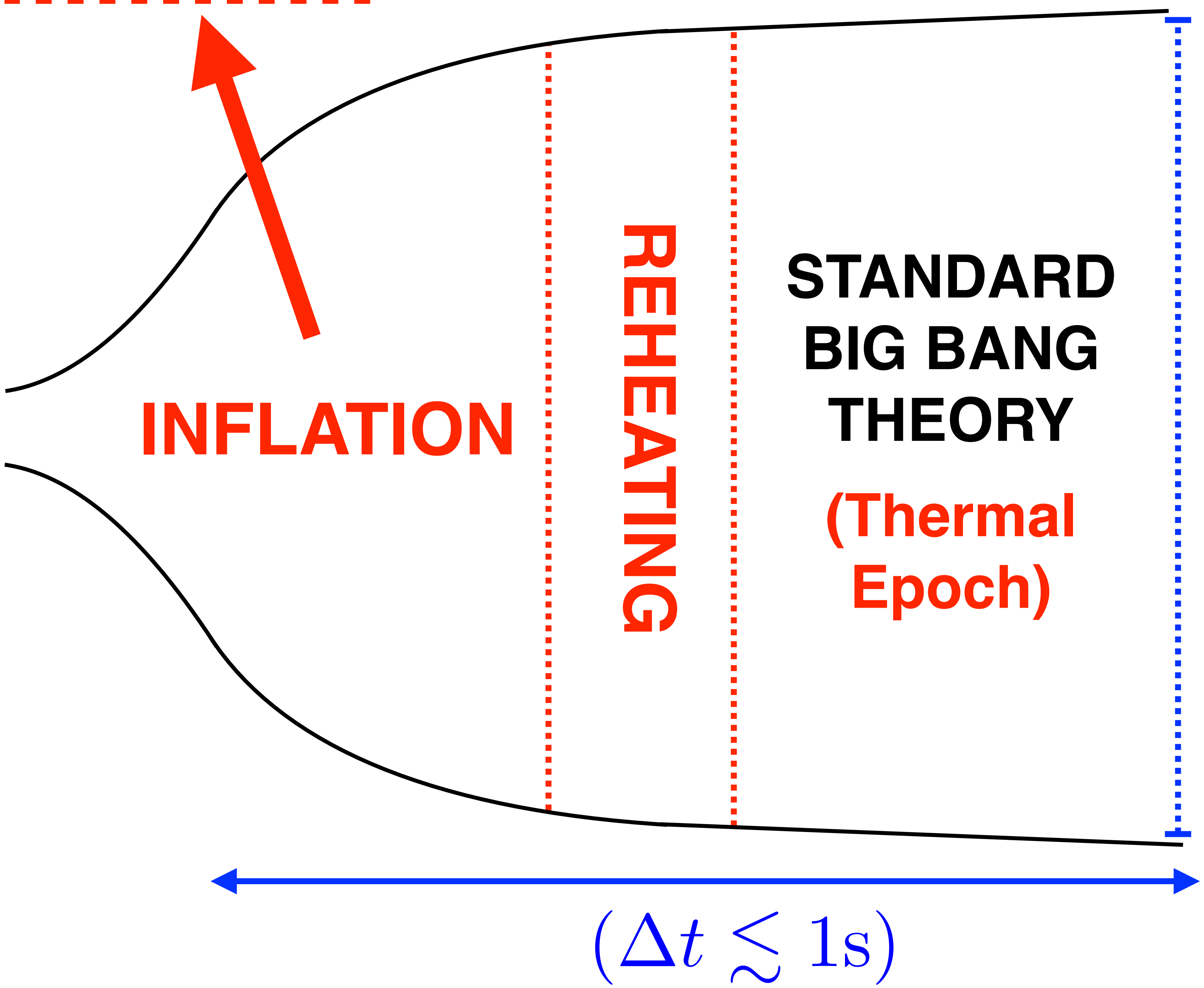
MOTIVATION

The Early Universe

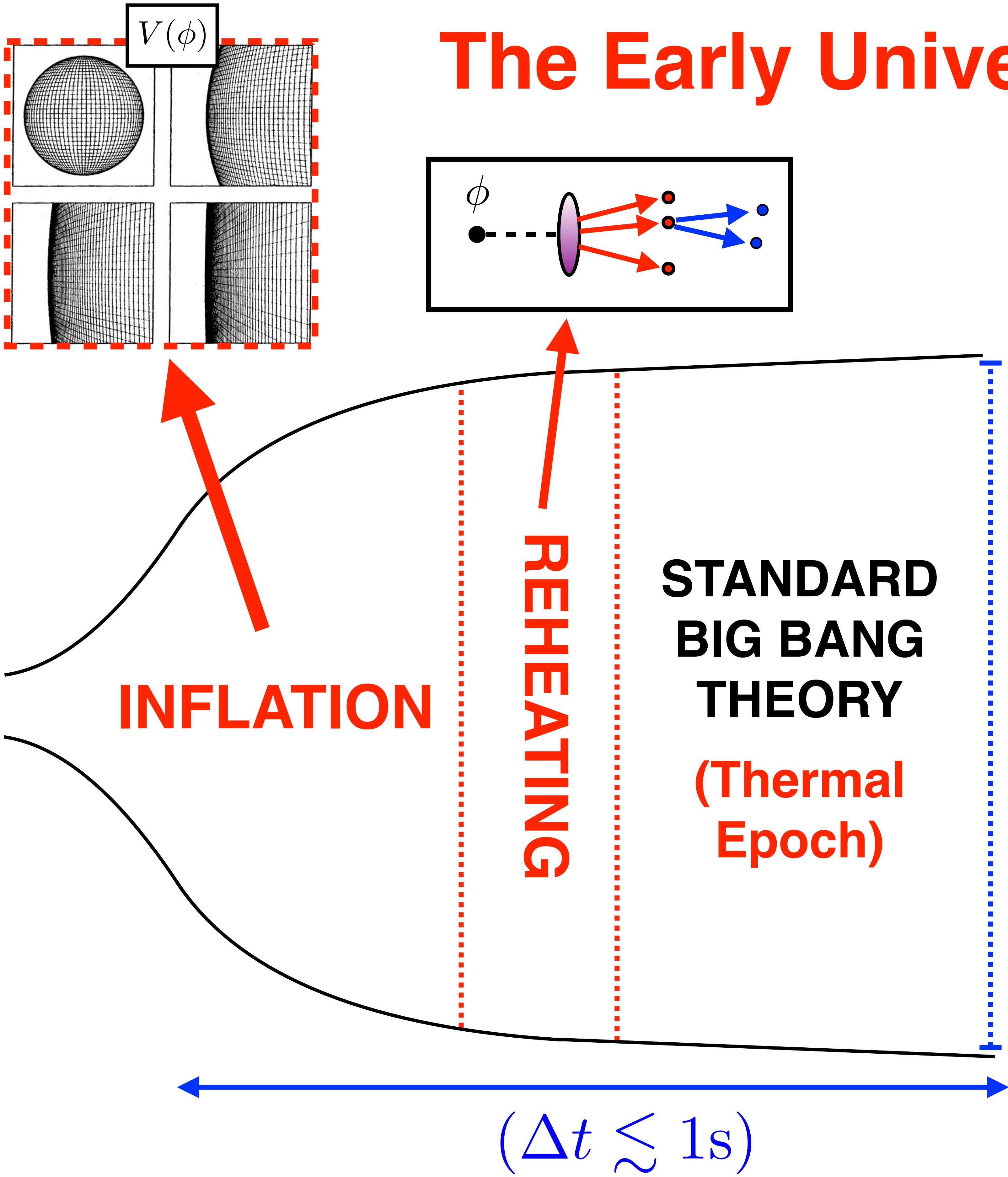


The Early Universe

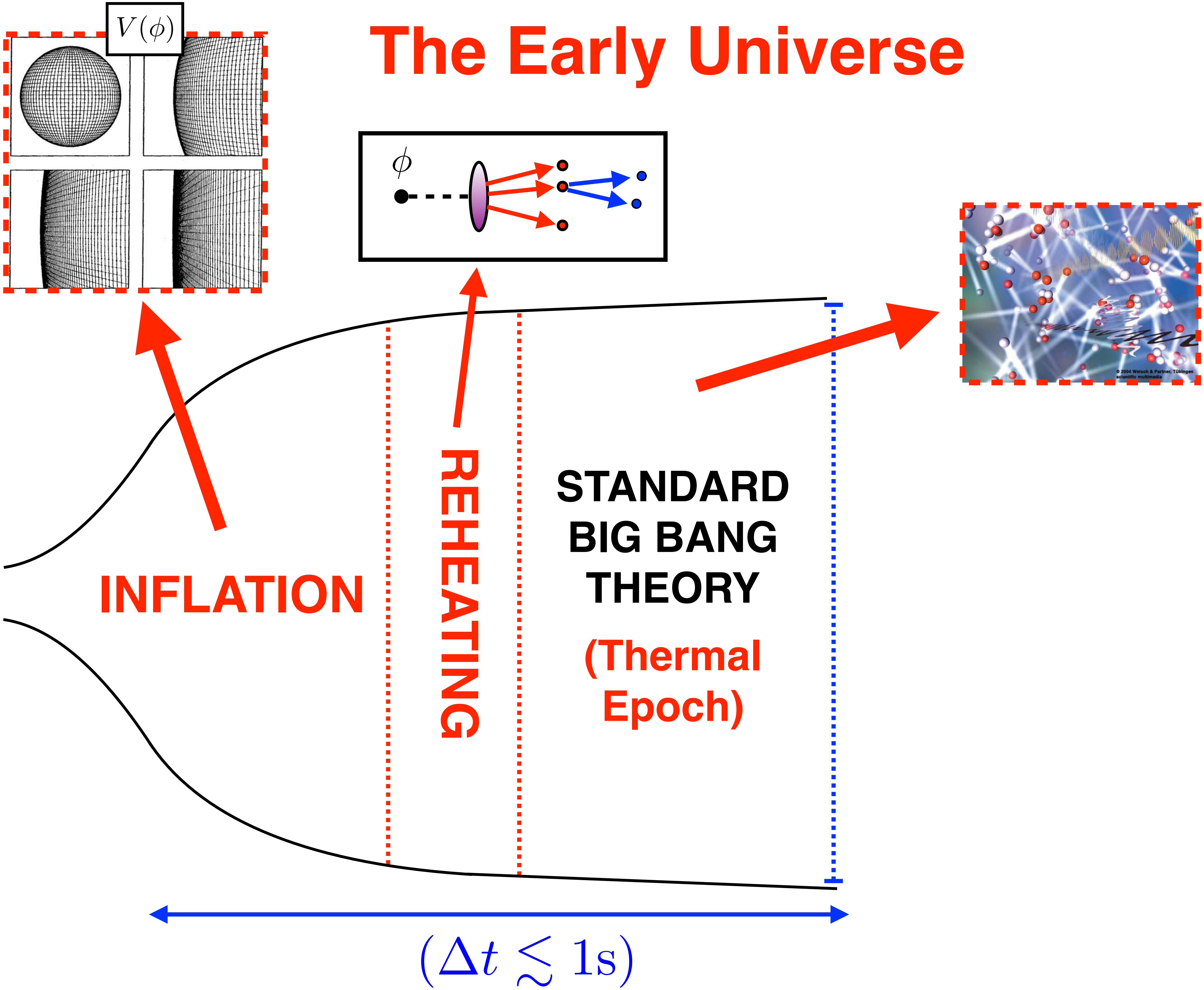




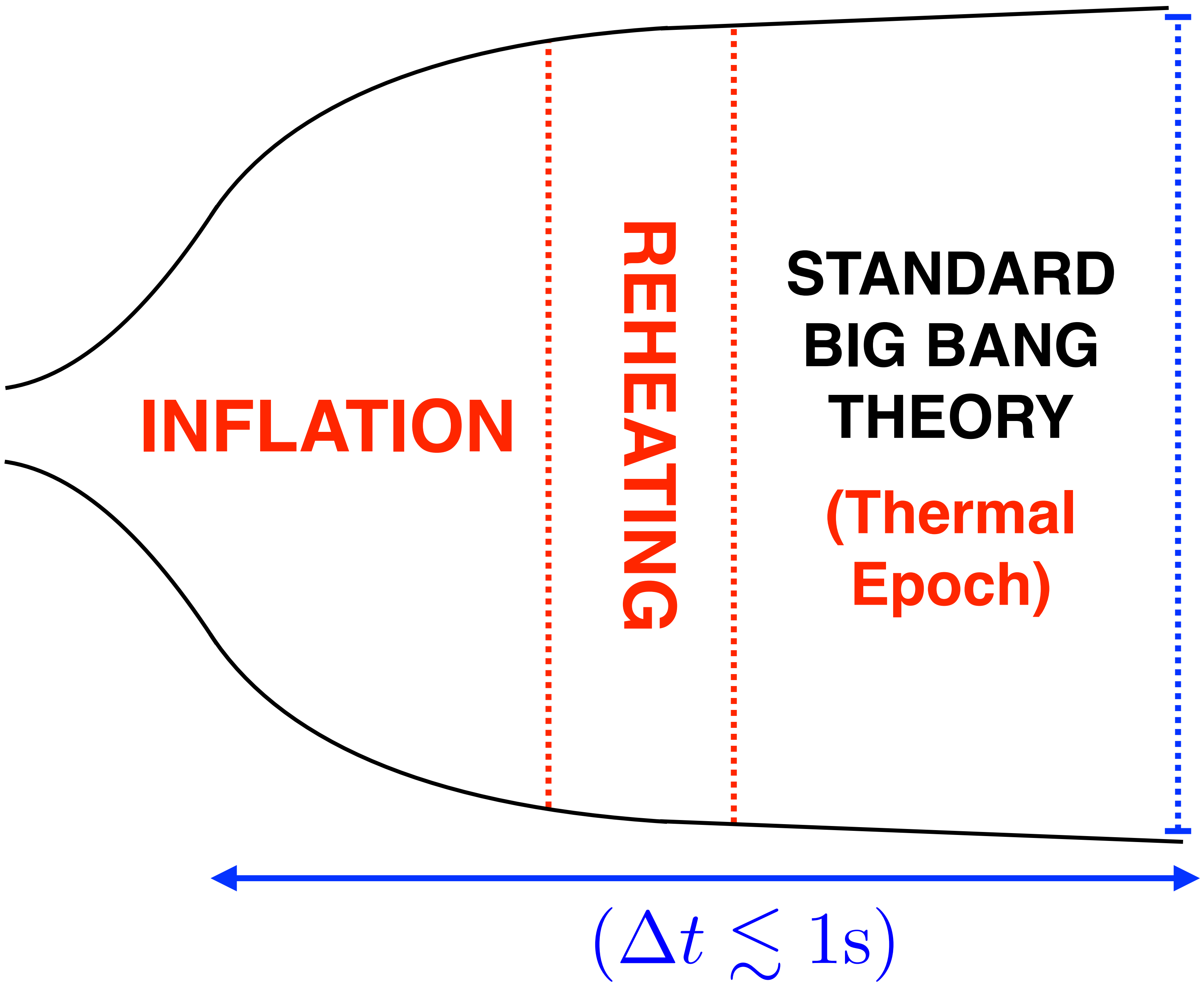
The Early Universe

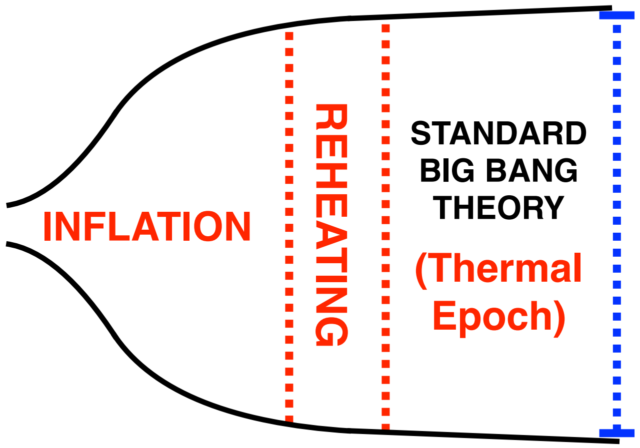


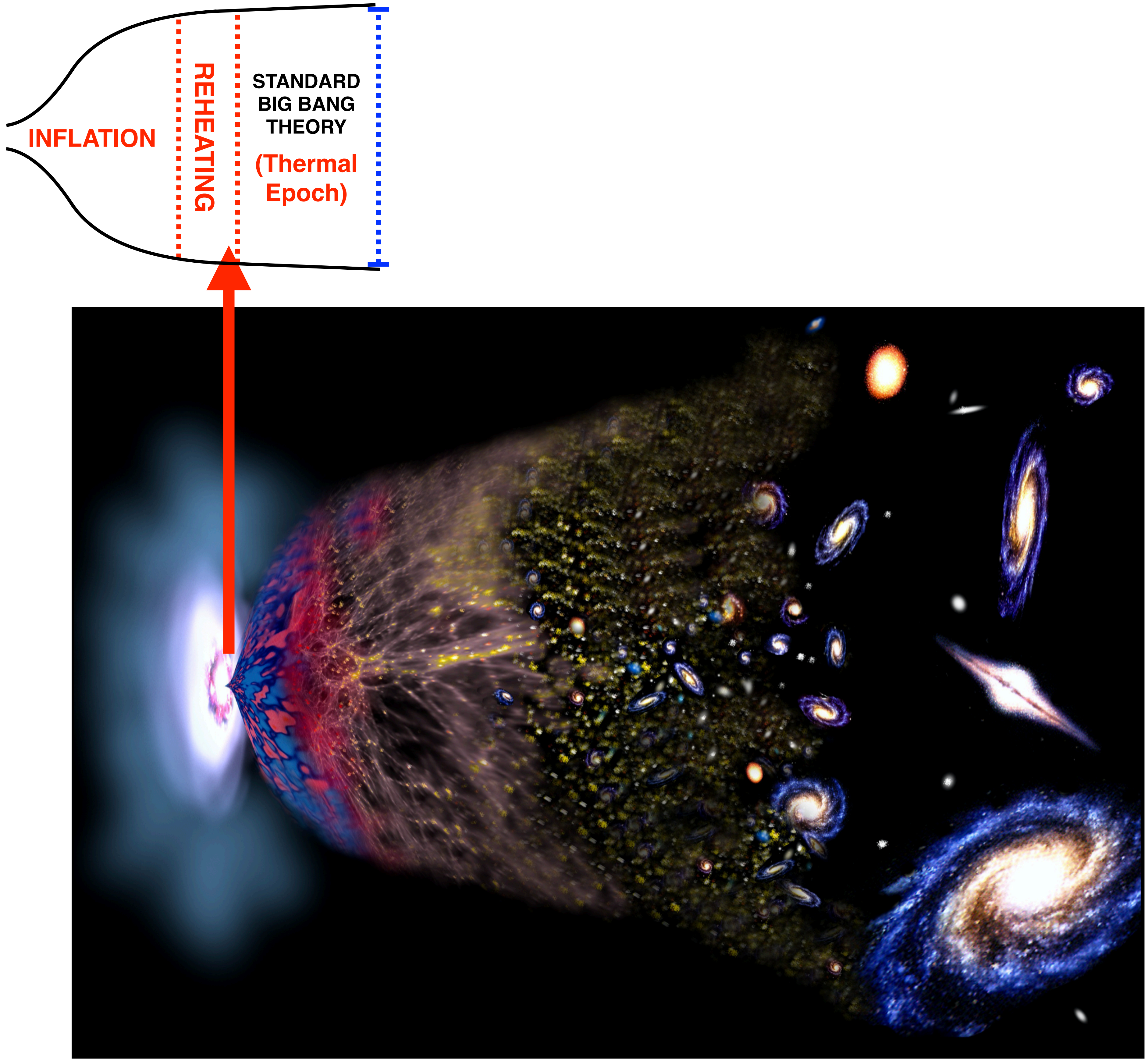
The Early Universe



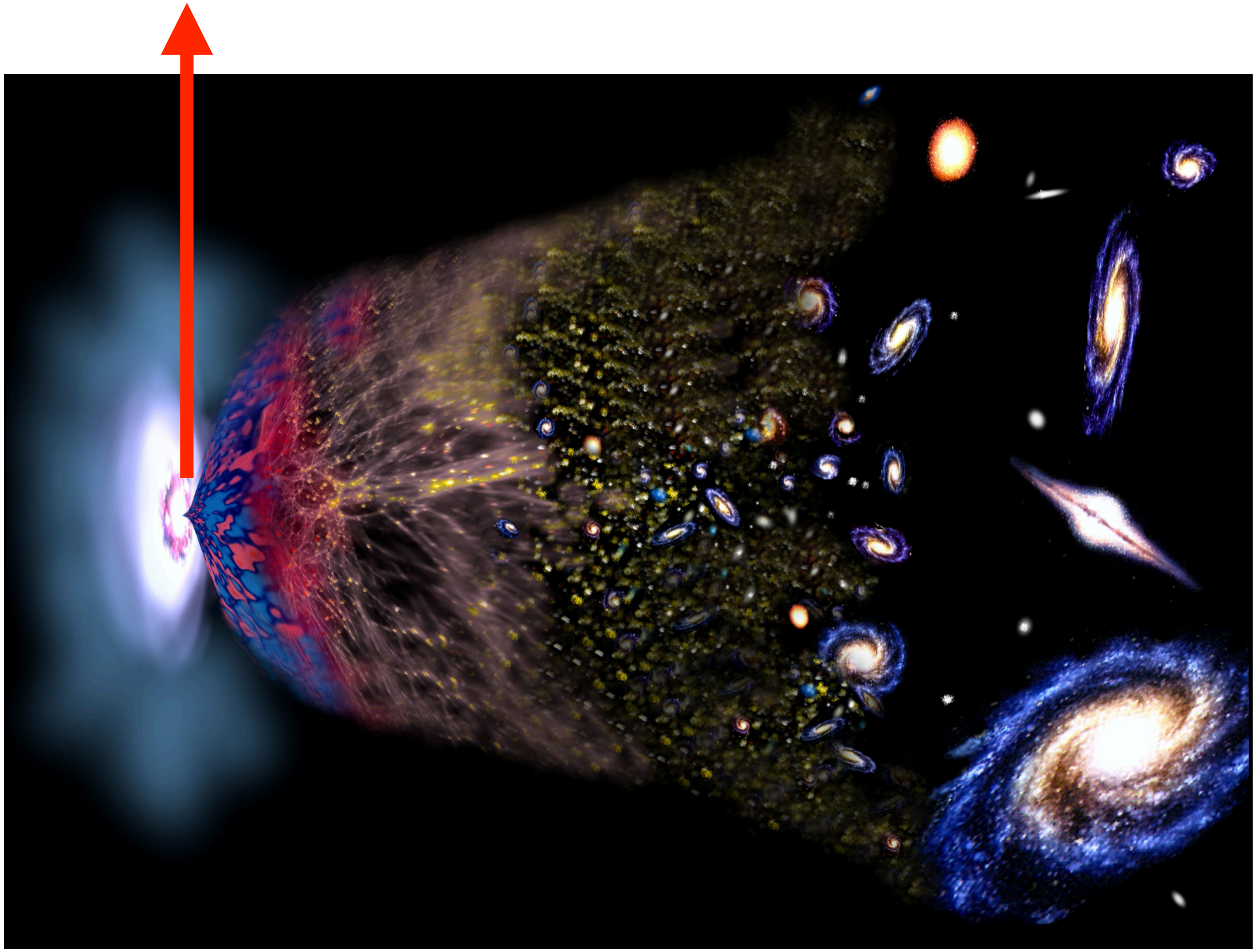
The Early Universe



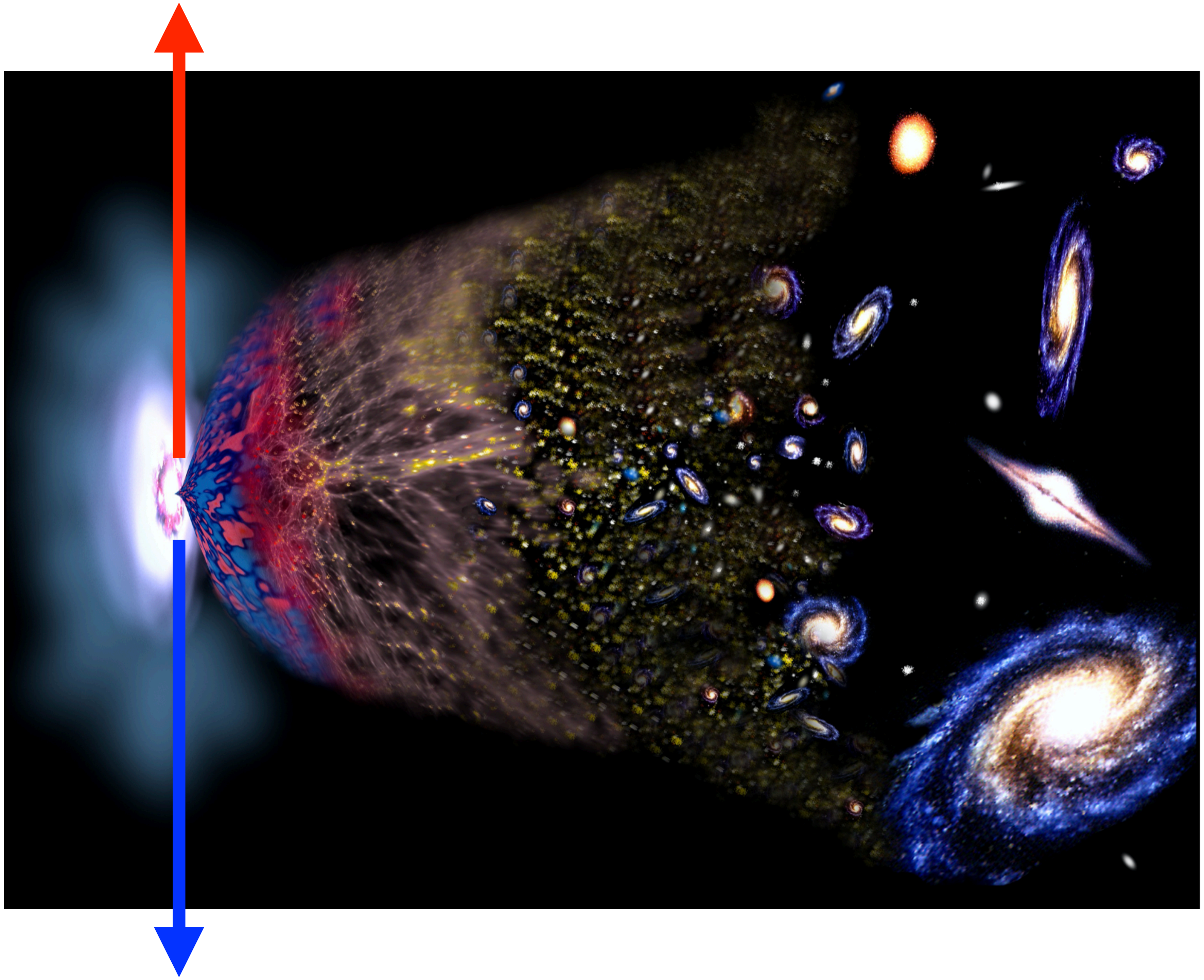

INFLATION
REHEATING
STANDARD
BIG BANG
THEORY
(Thermal
Epoch)


INFLATION
REHEATING
STANDARD
BIG BANG
THEORY
(Thermal
Epoch)

Can we simulate the very first instants of the Universe ?

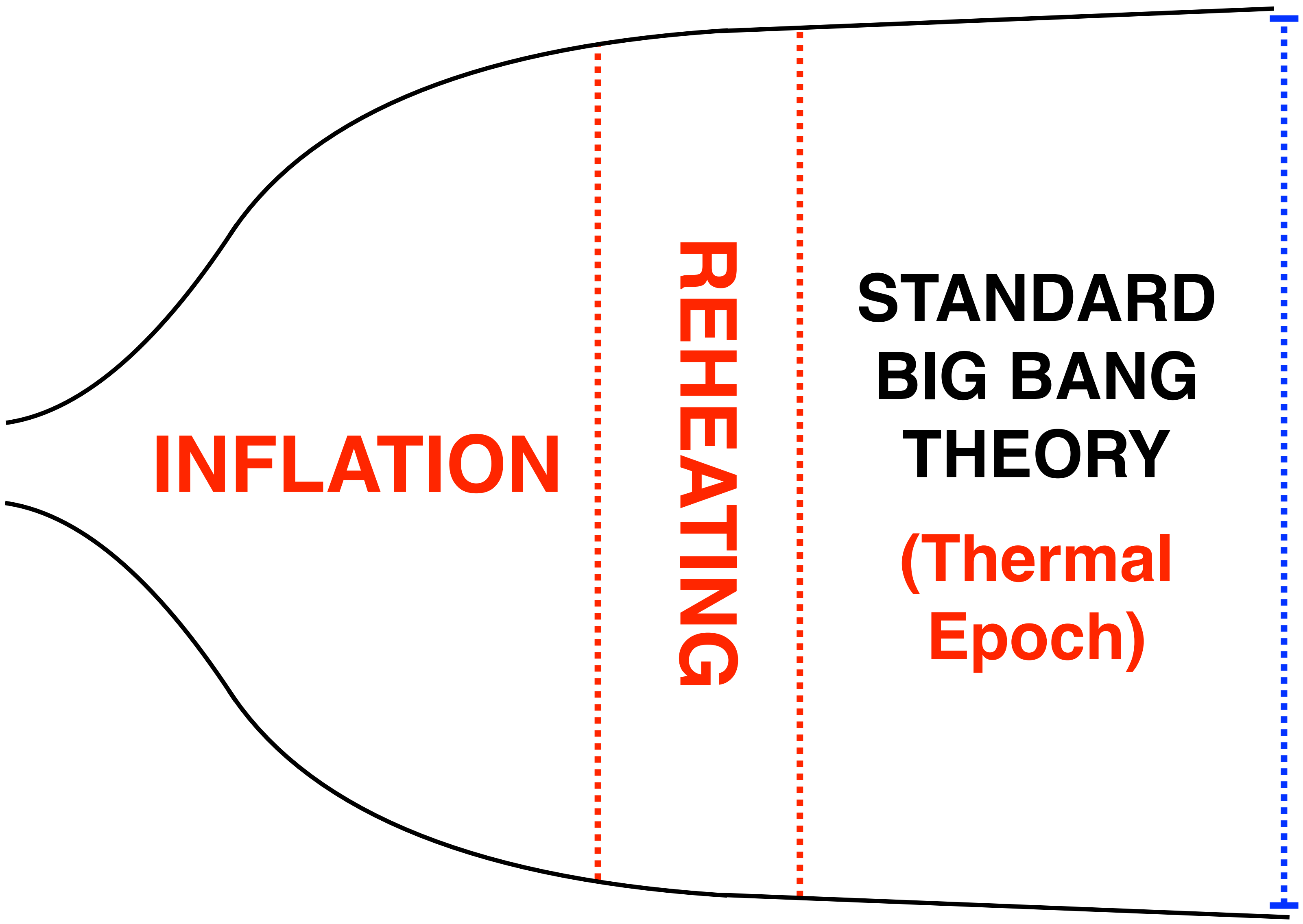


Can we simulate the very first instants of the Universe ?

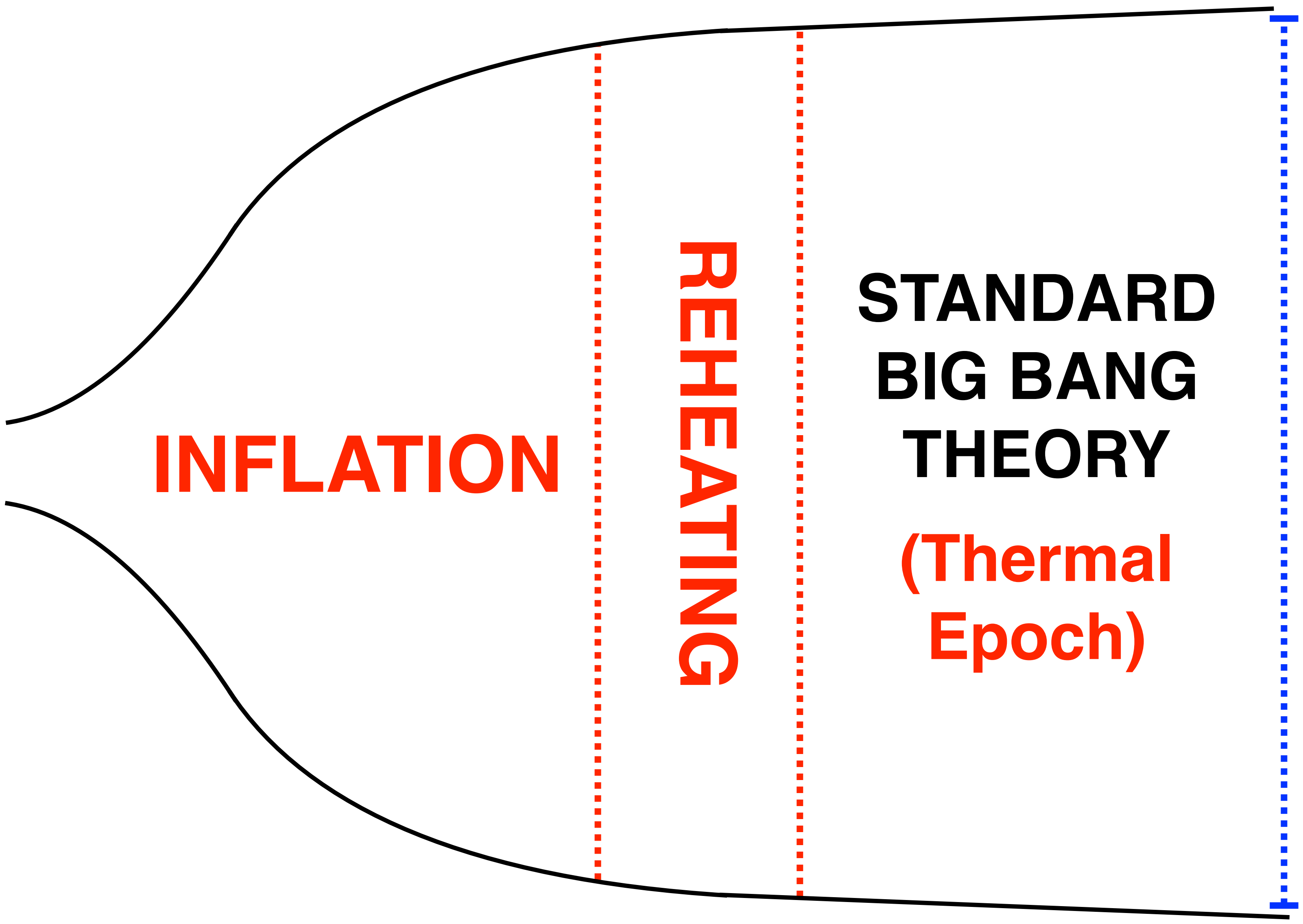


If so, How? What can we learn?

The Early Universe

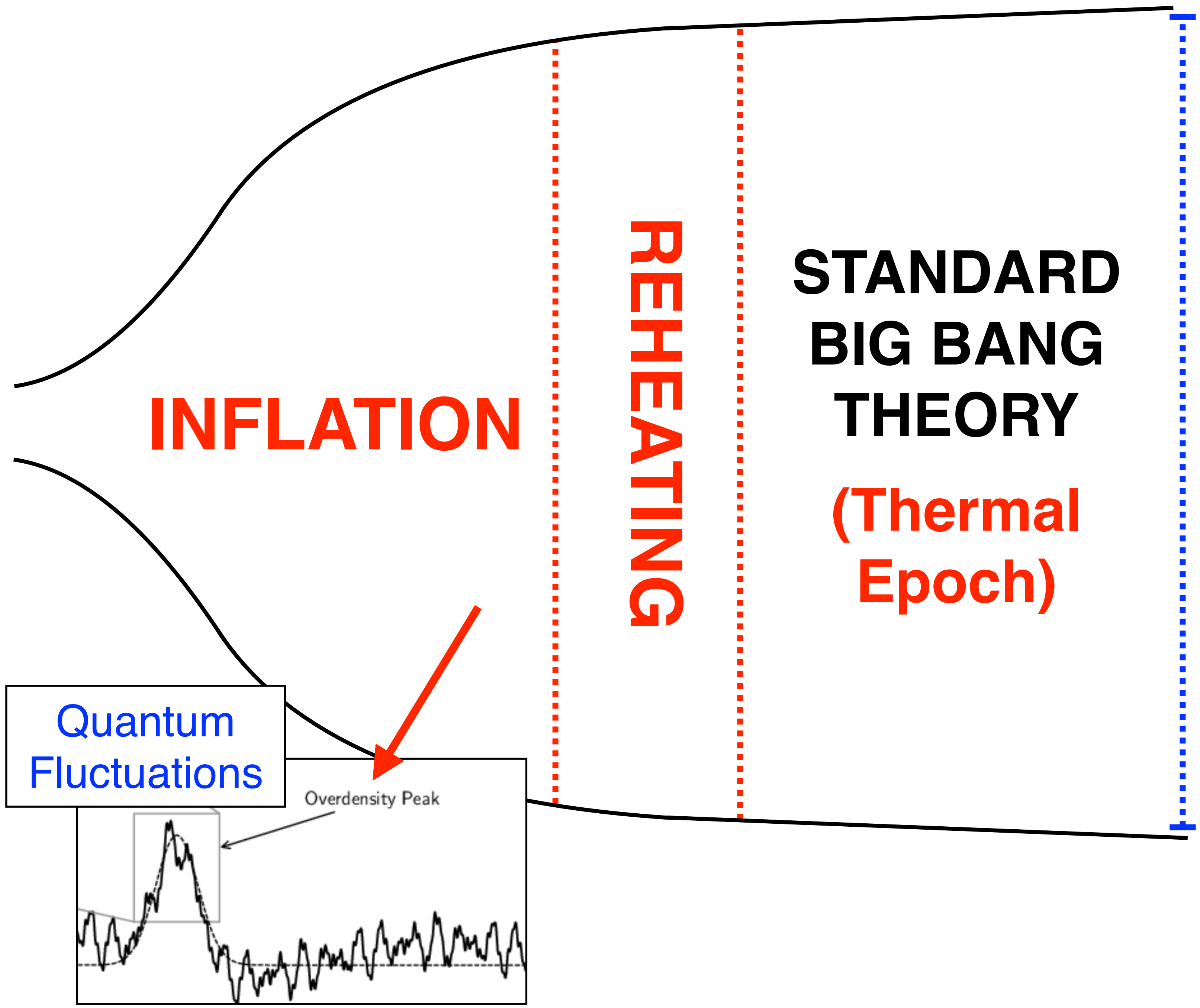


The Early Universe

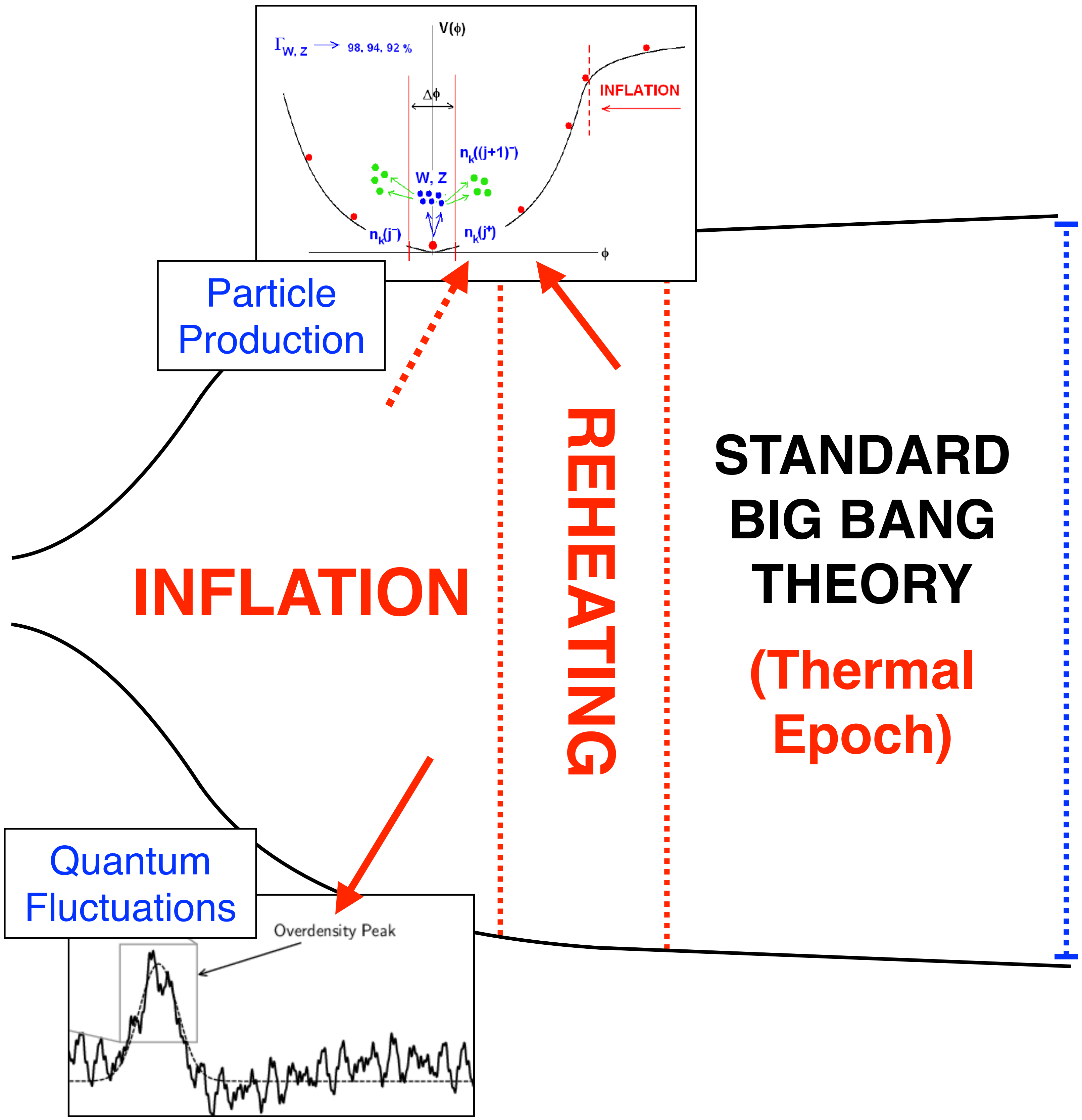


What interesting phenomena?

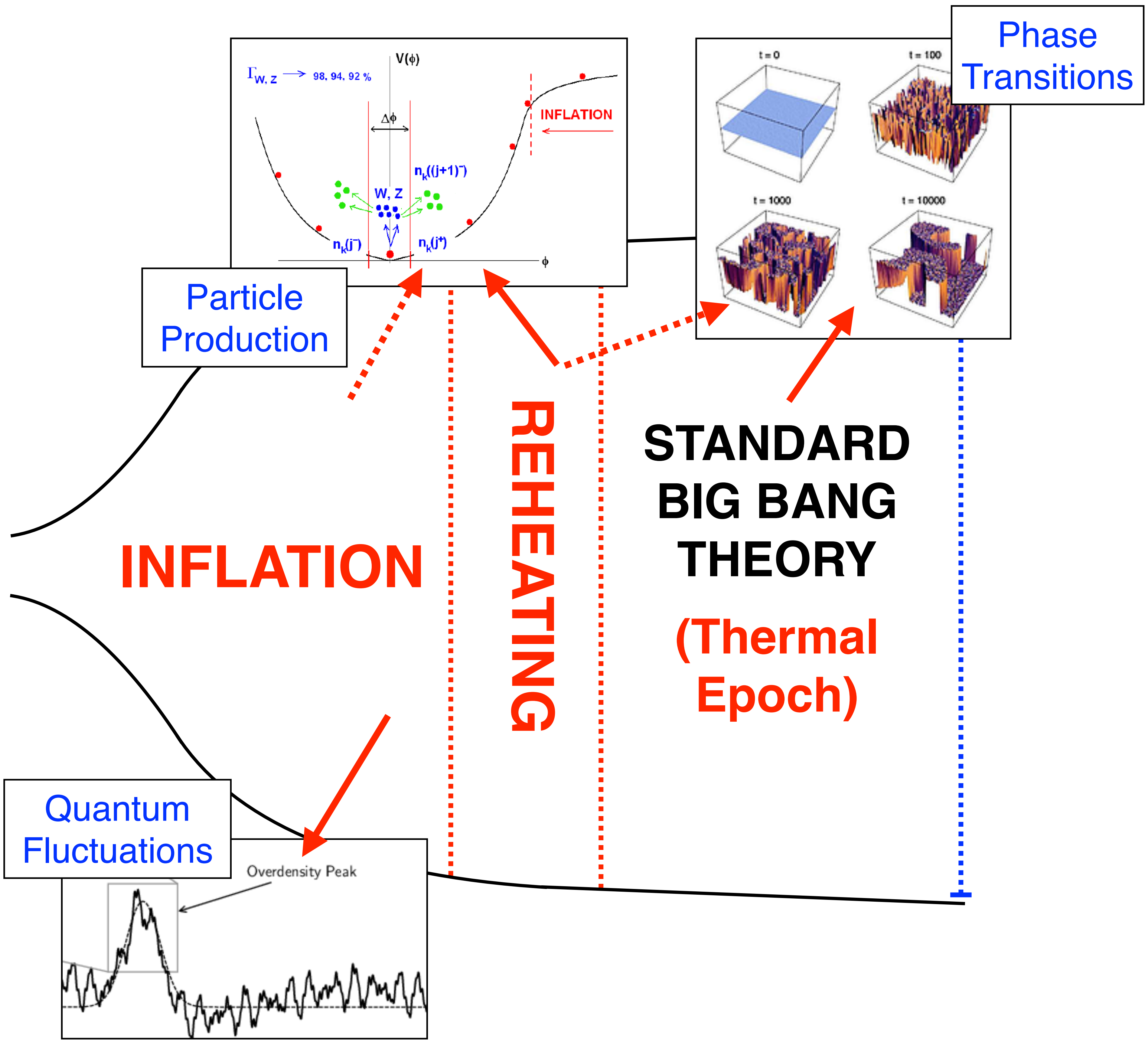
The Early Universe



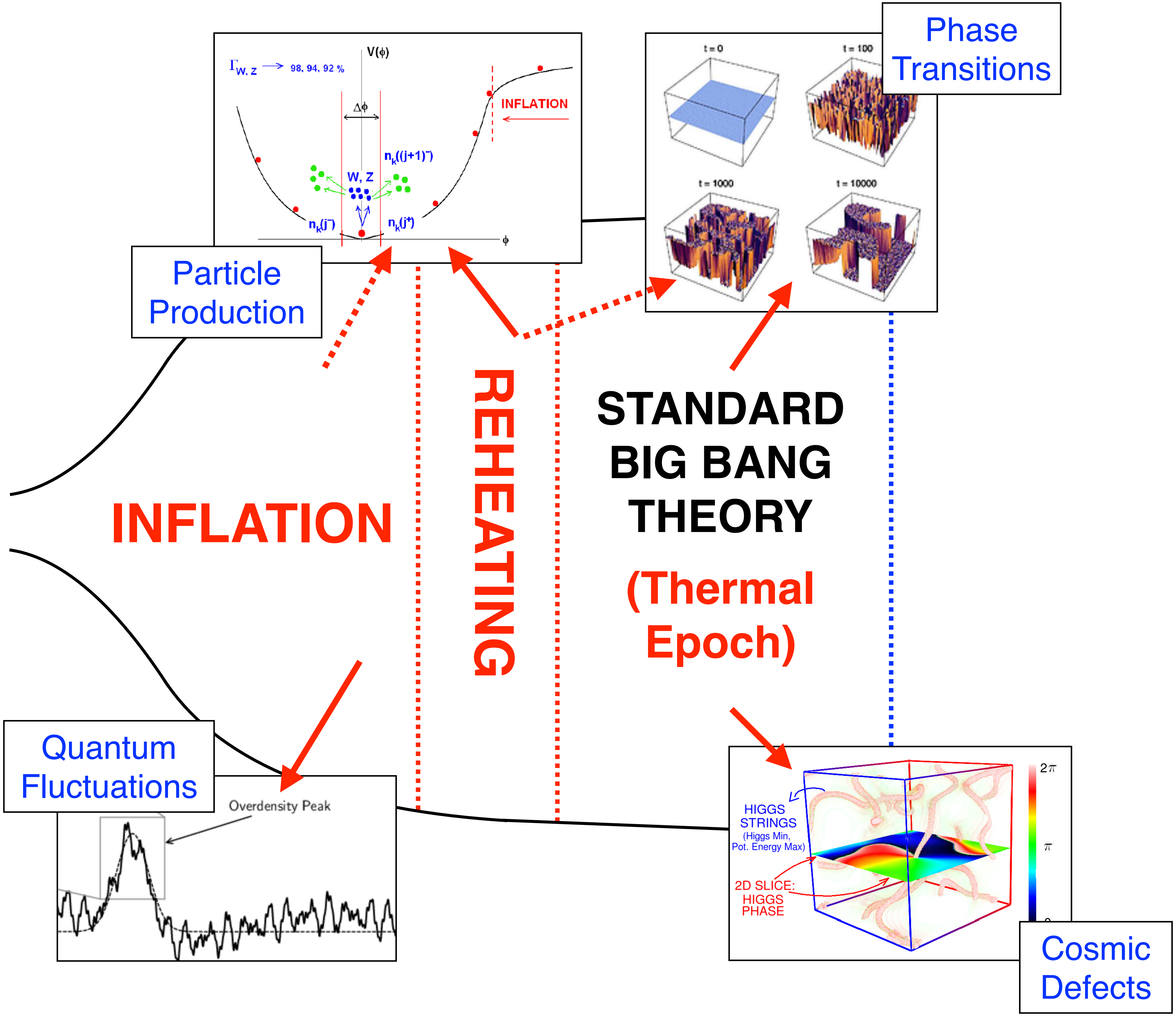
The Early Universe



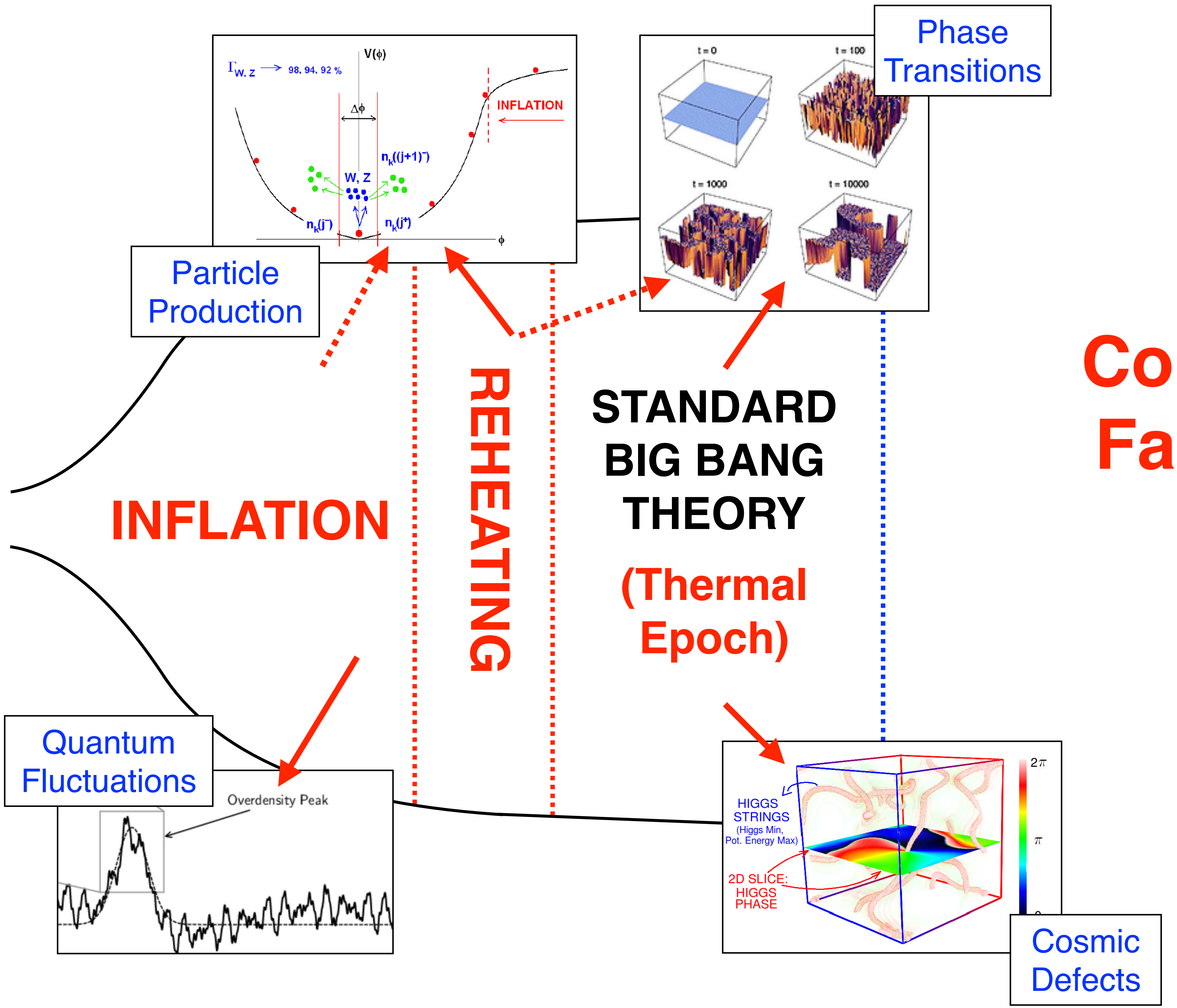
The Early Universe



The Early Universe

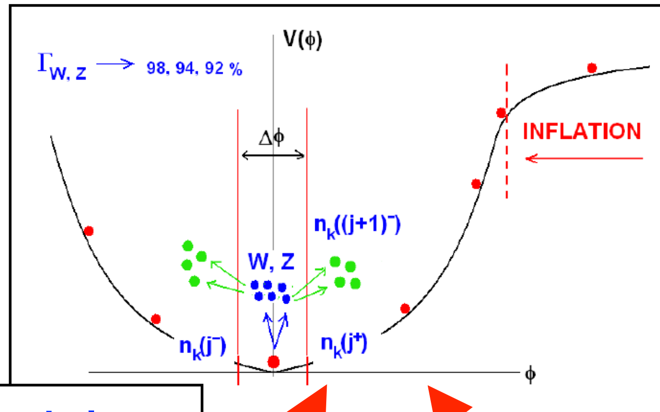


The Early Universe

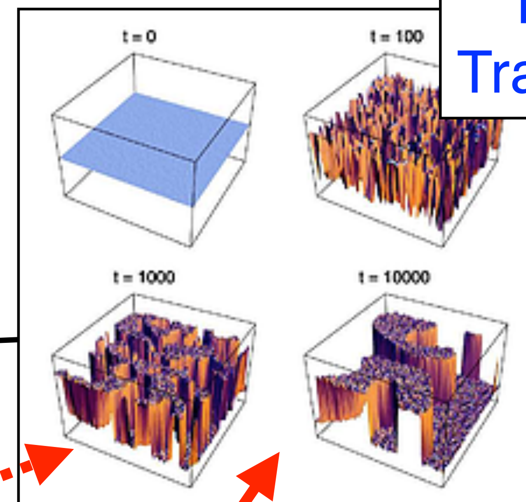


Common Factor ?

The Early Universe



Particle Production



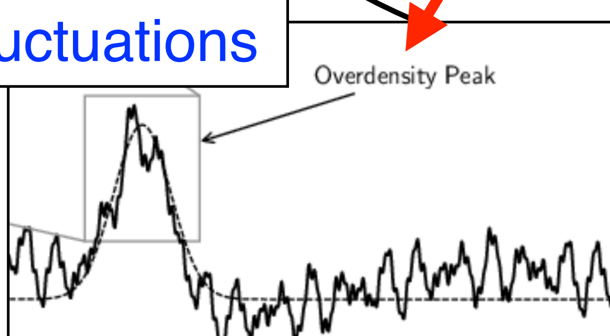
Phase Transitions

INFLATION

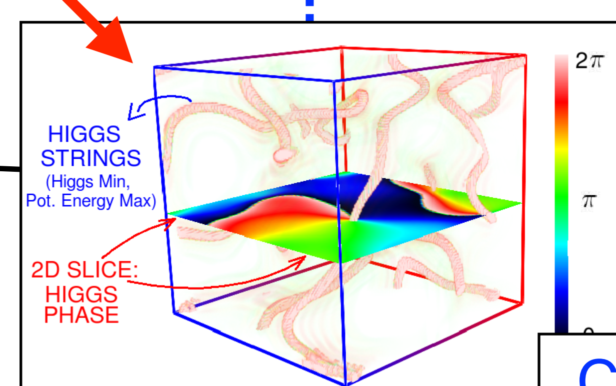
REHEATING

STANDARD BIG BANG THEORY

(Thermal Epoch)



Quantum Fluctuations



Cosmic Defects

Common Factor ?

Non-linear field dynamics

The Early Universe

Particle Production

Phase Transitions

Curvature Fluctuations

Cosmic Defects

Common Factor ?

Non-linear field dynamics

The Early Universe

Particle Production

Phase Transitions

Non-linear field dynamics

Curvature Fluctuations

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Baryo-genesis

Non-linear field dynamics

Magneto-genesis

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Non-linear field dynamics

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation ....

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Non-linear field dynamics

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation ….

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

Non-linear

≡ Numerical simulations

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation ....

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

The Early Universe

Gravitational waves

Particle Production

Phase Transitions

Non-minimal Gravitational Coupling

Baryo-genesis

CosmoLattice

Non-minimal Kinetic Theories

Magneto-genesis

Turbulence Thermalisation ....

Curvature Fluctuations

Black Hole Formation

Cosmic Defects

CosmoLattice – School 2022

School aimed to provide a pedagogical introduction to lattice field theory techniques

+

their adaptation to simulate the dynamics of interacting fields in an expanding background

CosmoLattice – School 2022

We will introduce *CosmoLattice*, our public code for lattice simulations of early Universe scenarios

… so you learn how to simulate non-linear scalar and gauge field dynamics in an expanding universe

CosmoLattice – School 2022

Day 1
(Monday 5th)

Day 2
(Tuesday 6th)

Day 3
(Wednesday 7th)

Day 4
(Thursday 8th)

CosmoLattice – School 2022

Day 1
(Monday 5th)

- **Lesson 1: What is a Lattice?**
- **Lesson 2: Inflation and post-inflationary dynamics**
- **Lesson 2b: Primer on Lattice simulations**
- **Practice**

Day 2
(Tuesday 6th)

- **Lesson 3: Evolution algorithms**
- **Lesson 4: Interacting scalar fields in an expanding background**
- **Topical 1: Gravitational non-minimally coupled scalar fields**
- **Practice**

Day 3
(Wednesday 7th)

- **Topical 2: Gravitational waves**
- **Practice**
- **Lesson 5: Lattice U(1) gauge theories**
- **Lesson 6: Lattice SU(2) gauge theories**

Day 4
(Thursday 8th)

- **Topical 3: Non-linear dynamics of axion inflation**
- **Lesson 7: Parallelization techniques in CosmoLattice**
- **Topical 4: Plotting 3D data with CosmoLattice**
- **Overview + Practice**

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- **Topical 3: Non-linear dynamics of axion inflation**
- **Lesson 7: Parallelization techniques in CosmoLattice**
- **Topical 4: Plotting 3D data with CosmoLattice**
- **Overview + Practice**

CosmoLattice – School 2022

Day 1 (Monday 5th)
- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2 (Tuesday 6th)
- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3 (Wednesday 7th)
- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4 (Thursday 8th)
- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

Basic

Advanced

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields

Day 3
(Wednesday 7th)

- Topical 2: Gravitational waves
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview

CosmoLattice – School 2022

Day 1
(Monday 5th)

Lesson 1: What is a Lattice?
Lesson 2: Inflation and post-inflationary dynamics
Lesson 2b: Primer on Lattice simulations

Day 2
(Tuesday 6th)

Lesson 3: Evolution algorithms
Lesson 4: Interacting scalar fields in an expanding background
Topical 1: Gravitational non-minimally coupled scalar fields

Day 3
(Wednesday 7th)

Topical 2: Gravitational waves

Lesson 5: Lattice U(1) gauge theories
Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

Topical 3: Non-linear dynamics of axion inflation
Lesson 7: Parallelization techniques in CosmoLattice
Topical 4: Plotting 3D data with CosmoLattice
Overview

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- **Practice**

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- **Practice**

Day 3
(Wednesday 7th)

- **Practice**
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + **Practice**

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3
(Wednesday 7th)

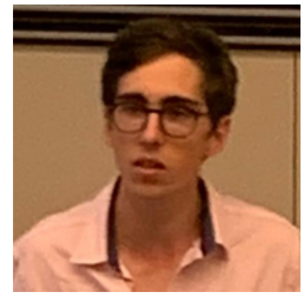
- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

CosmoLattice – School 2022

Daniel FIGUEROA
(IFIC, Valencia, Spain)



Adrien FLORIO
(Stony Brook, USA)

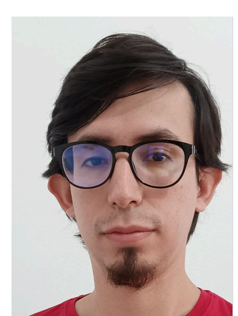
Francisco TORRENTI
(Basel Univ., Switzerland)

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



Nicolás LOAYZA
(IFIC, Valencia, Spain)

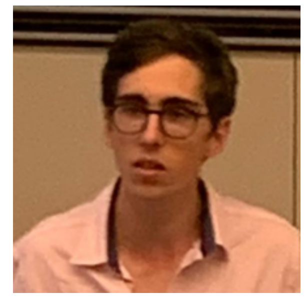
Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

CosmoLattice – School 2022

CosmoLattice creators and main lecturers

Daniel FIGUEROA
(IFIC, Valencia, Spain)



Adrien FLORIO
(Stony Brook, USA)

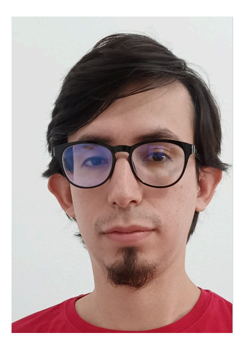
Francisco TORRENTI
(Basel Univ., Switzerland)

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



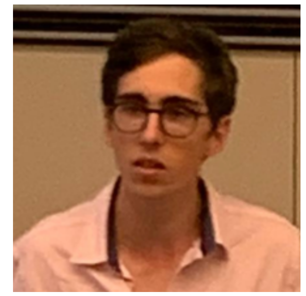
Nicolás LOAYZA
(IFIC, Valencia, Spain)

Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

CosmoLattice – School 2022

Daniel FIGUEROA
(IFIC, Valencia, Spain)



Adrien FLORIO
(Stony Brook, USA)

Francisco TORRENTI
(Basel Univ., Switzerland)

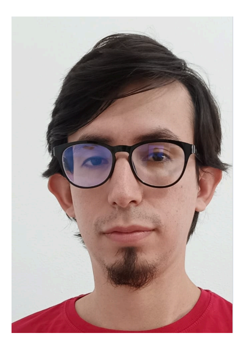
CosmoLattice contributors and topical lecturers

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



Nicolás LOAYZA
(IFIC, Valencia, Spain)

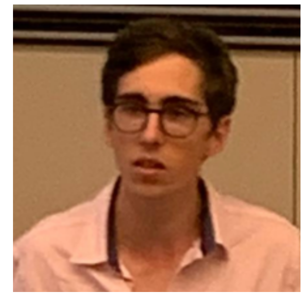
Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

CosmoLattice – School 2022

Faculty

Daniel FIGUEROA
(IFIC, Valencia, Spain)



Adrien FLORIO
(Stony Brook, USA)

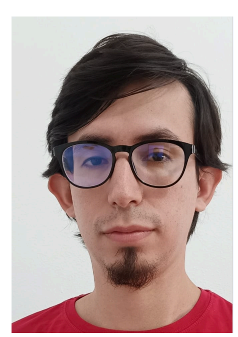
Francisco TORRENTI
(Basel Univ., Switzerland)

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



Nicolás LOAYZA
(IFIC, Valencia, Spain)

Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

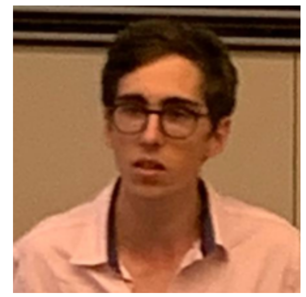
CosmoLattice – School 2022

Faculty

Daniel FIGUEROA
(IFIC, Valencia, Spain)

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Postdocs



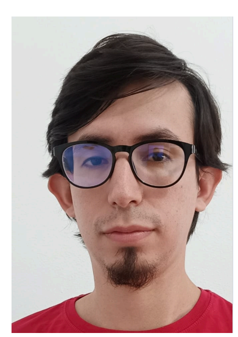
Adrien FLORIO
(Stony Brook, USA)

Francisco TORRENTI
(Basel Univ., Switzerland)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



Nicolás LOAYZA
(IFIC, Valencia, Spain)

Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

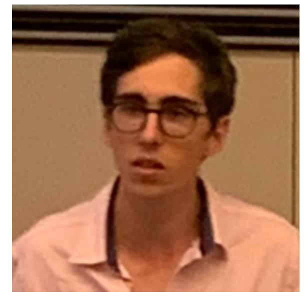
CosmoLattice – School 2022

Faculty

Daniel FIGUEROA
(IFIC, Valencia, Spain)

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Postdocs



Adrien FLORIO
(Stony Brook, USA)

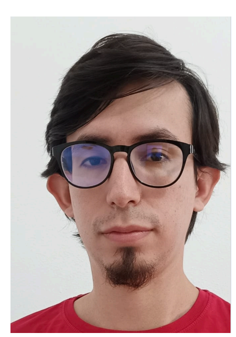
Francisco TORRENTI
(Basel Univ., Switzerland)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

PhD

Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)



Nicolás LOAYZA
(IFIC, Valencia, Spain)

Ken Marschall
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

Day 2
(Tuesday 6th)

- Lesson 3: Evolution algorithms
- Lesson 4: Interacting scalar fields in an expanding background
- Topical 1: Gravitational non-minimally coupled scalar fields
- Practice

Day 3
(Wednesday 7th)

- Topical 2: Gravitational waves
- Practice
- Lesson 5: Lattice U(1) gauge theories
- Lesson 6: Lattice SU(2) gauge theories

Day 4
(Thursday 8th)

- Topical 3: Non-linear dynamics of axion inflation
- Lesson 7: Parallelization techniques in CosmoLattice
- Topical 4: Plotting 3D data with CosmoLattice
- Overview + Practice

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: What is a Lattice?
- Lesson 2: Inflation and post-inflationary dynamics
- Lesson 2b: Primer on Lattice simulations
- Practice

CosmoLattice – School 2022

Day 1
(Monday 5th)

- **Lesson 1: What is a Lattice? — Dani**
- **Lesson 2: Inflation and post-inflationary dynamics — Paco**
- **Lesson 2b: Primer on Lattice simulations — Paco**
- **Practice — All together**

CosmoLattice – School 2022

Day 1
(Monday 5th)

- **Lesson 1: What is a Lattice? — Dani**
- **Lesson 2: Inflation and post-inflationary dynamics — Paco**
- **Lesson 2b: Primer on Lattice simulations — Paco**
- **Practice — All together**

CosmoLattice – School 2022

— Lecture 1 —

Welcome to the Lattice

Daniel G. Figueroa
IFIC UV/CSIC, Spain

Adrien Florio
Stony Brook U., USA

Francisco Torrenti
U. Basel, Switzerland

CosmoLattice – School 2022

— Lecture 1 —
Welcome to the Lattice

Daniel G. Figueroa
IFIC UV/CSIC, Spain

Adrien Florio
Stony Brook U., USA

Francisco Torrenti
U. Basel, Switzerland

CosmoLattice – School 2022

— Lecture 1 —

Welcome to the Lattice

* L1.a: Overview of CosmoLattice (CL)
* L1.b: What is really a Lattice ?

Daniel G. Figueroa
IFIC UV/CSIC, Spain

Adrien Florio
Stony Brook U., USA

Francisco Torrenti
U. Basel, Switzerland

CosmoLattice – School 2022

— Lecture 1 —

Welcome to the Lattice

* L1.a: Overview of CosmoLattice (CL)
* L1.b: What is really a Lattice ?

Daniel G. Figueroa
IFIC UV/CSIC, Spain

Adrien Florio
Stony Brook U., USA

Francisco Torrenti
U. Basel, Switzerland

— Lecture 1.a —

Overview of CosmoLattice (CL)

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

http://www.cosmolattice.net/

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]

Lecture 4 **Lectures 5 & 6**

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]

Lecture 4 **Lectures 5 & 6**

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]

- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]

- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).

Lecture 2

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]
- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).
- **Parallellized** in multiple spatial dimensions (**but you write in serial !**)

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]
- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).
- **Parallellized** in multiple spatial dimensions (**but you write in serial !**) Lecture 7

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]
- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).
- **Parallellized** in multiple spatial dimensions (**but you write in serial**)
- **Family** of evolution **algorithms**, accuracy ranging from $\delta\mathcal{O}(\delta t^2) - \delta\mathcal{O}(\delta t^{10})$

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

- Simulates **scalar** and **gauge field dynamics** [including **U(1) & SU(2)** interactions]
- Written in **C++**, with **modular structure** separating physics (CosmoInterface library) and technical details (TempLat library).
- **Parallellized** in multiple spatial dimensions (**but you write in serial**)
- **Family** of evolution **algorithms**, accuracy ranging from $\delta\mathcal{O}(\delta t^2)$ – $\delta\mathcal{O}(\delta t^{10})$

Lecture 3 ⟶ Lectures 4, 5 & 6

CosmoLattice

Figueroa, Florio, Torrenti, Valkenburg

Lattice Theory Review: arXiv: 2006.15122

Code Manual: arXiv: 2102.01031

http://www.cosmolattice.net/

What Field theory ?

- **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

What Field theory ?

- **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

$$\phi \in \mathscr{R}e$$

Scalar sector

What Field theory ?

➤ **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \underbrace{(D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

$$\phi \in \mathscr{R}e$$

Scalar sector

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$

$$D_\mu^{\mathrm{A}} \equiv \partial_\mu - iQ_A^{(\varphi)}g_{_A}A_\mu$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

U(1) gauge sector

What Field theory ?

- **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

$$\phi \in \mathscr{R}e$$

Scalar sector

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$

$$D_\mu^{\mathrm{A}} \equiv \partial_\mu - iQ_{\mathrm{A}}^{(\varphi)}g_{_A}A_\mu$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

U(1) gauge sector

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1\\ \varphi_2 + i\varphi_3\end{pmatrix}$$

$$D_\mu \equiv \mathscr{I}D_\mu^{\mathrm{A}} - ig_BQ_BB_\mu^a\,T_a$$

$$G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]$$

SU(2) gauge sector

What Field theory ?

➤ **Matter content:**

$$S = -\int d^4x\sqrt{-g}\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D^A_\mu\varphi)^*(D^\mu_A\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

$$\phi \in \mathscr{R}e$$

Scalar sector

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$

$$D^{\mathrm{A}}_\mu \equiv \partial_\mu - iQ^{(\varphi)}_A g_{_A} A_\mu$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

U(1) gauge sector

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1\\ \varphi_2 + i\varphi_3\end{pmatrix}$$

$$D_\mu \equiv \mathcal{I} D^{\mathrm{A}}_\mu - ig_B Q_B B^a_\mu T_a$$

$$G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]$$

SU(2) gauge sector

Scalar potential

What Field theory ?

- **Matter content:**

$$S = -\int d^4x\sqrt{-g}\,\left\{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + (D_\mu^A\varphi)^*(D_A^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}\{G_{\mu\nu}G^{\mu\nu}\} + V(\phi,|\varphi|,|\Phi|)\right\}$$

$$\phi \in \mathscr{R}e$$

Scalar sector

$$\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_0 + i\varphi_1)$$
$$D_\mu^{\rm A} \equiv \partial_\mu - iQ_A^{(\varphi)}g_{_A}A_\mu$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

U(1) gauge sector

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1\\ \varphi_2 + i\varphi_3\end{pmatrix}$$
$$D_\mu \equiv \mathscr{I}D_\mu^{\rm A} - ig_BQ_BB_\mu^a\,T_a$$
$$G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]$$

SU(2) gauge sector

Scalar potential

- **Background Metric:**

$$ds^2 = -\,dt^2 + a^2(t)\,\delta_{ij}\,dx^idx^j$$

- **Self-consistent expansion** (Friedmann equations)
- **Fixed power-law background** $a(t) \sim t^{\frac{2}{3(1+w)}}$

Lattice Equations

- **Hamiltonian scheme**: coupled first-order differential equations

- **Scalar fld example**

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial \phi} \quad \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

KICK: $(\pi_\phi)' = -a^{3+\alpha}\frac{\partial V}{\partial \phi} + a^{1+\alpha}\nabla^2\phi$

DRIFT: $\phi' \equiv \pi_\phi a^{\alpha-3}$

Lattice Equations

➤ **Hamiltonian scheme**: coupled first-order differential equations

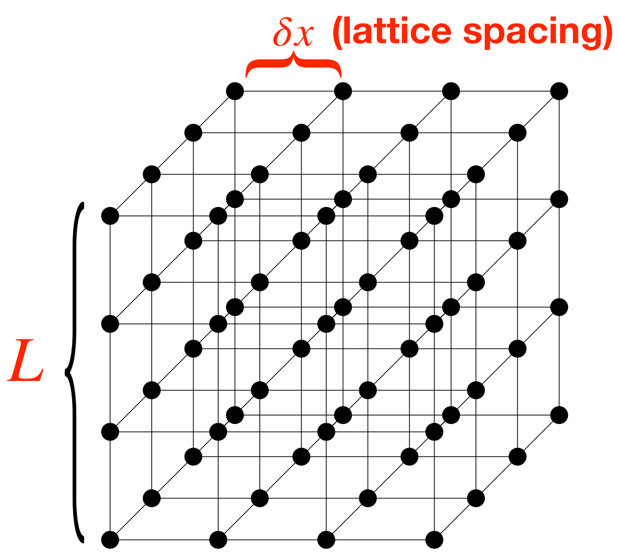
➤ **Scalar fld example**

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial \phi} \quad \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

KICK: $(\pi_\phi)' = -a^{3+\alpha}\frac{\partial V}{\partial \phi} + a^{1+\alpha}\nabla^2\phi$

DRIFT: $\phi' \equiv \pi_\phi a^{\alpha-3}$

➤ **Scalar Fields and momenta** are defined in the **lattice sites**



N : number of points/dimension

$L = N \cdot \delta x$: length side

δt : time step

Minimum and maximum momenta:

$$k_{\min} = \frac{2\pi}{L} \qquad k_{\max} = \frac{\sqrt{3}}{2} N k_{\min}$$

Lattice Equations

- **Hamiltonian scheme**: coupled first-order differential equations

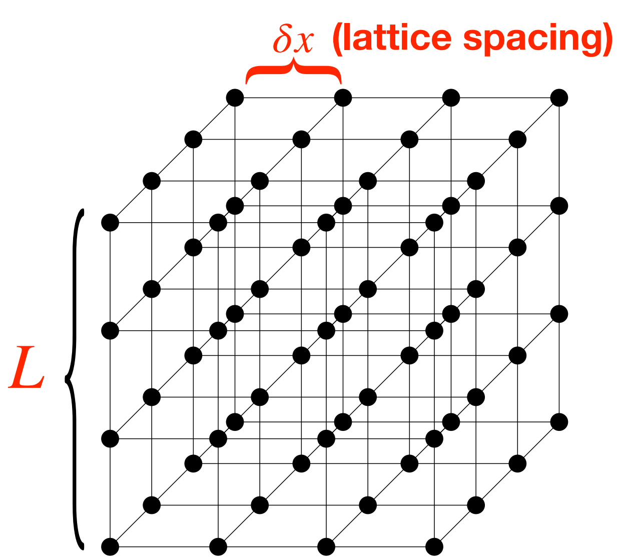
- **Scalar fld example**

$$\frac{d^2\phi}{dt^2} - \frac{1}{a^2}\nabla^2\phi + \frac{3}{a}\frac{da}{dt}\frac{d\phi}{dt} = -\frac{\partial V}{\partial\phi} \quad \xrightarrow{\pi_\phi \equiv \phi' a^{3-\alpha}}$$

KICK: $(\pi_\phi)' = -a^{3+\alpha}\frac{\partial V}{\partial \phi} + a^{1+\alpha}\nabla^2\phi$

DRIFT: $\phi' \equiv \pi_\phi a^{\alpha-3}$

- **Scalar Fields and momenta** are defined in the **lattice sites**



N : number of points/dimension

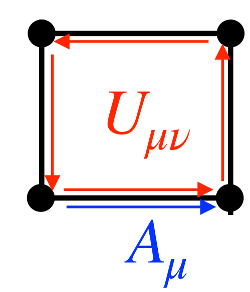
$L = N \cdot \delta x$: length side

δt : time step

Minimum and maximum momenta:

$$k_{\rm min} = \frac{2\pi}{L} \qquad k_{\rm max} = \frac{\sqrt{3}}{2} N k_{\rm min}$$

- **Gauge fields** introduced via **links** and **plaquettes** (like in **lattice-QCD**)



Writing a model

➤ Equations solved in (dimensionless) ***program variables*:**

Choose:

$$\{\alpha, \omega_*, f_*\}$$

→

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

$$\tilde{\phi} = \frac{\phi}{f_*} \qquad \tilde{\varphi} = \frac{\varphi}{f_*} \qquad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \qquad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

Writing a model

➤ Equations solved in (dimensionless) ***program variables***:

Choose:

$$\{\alpha, \omega_*, f_*\}$$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

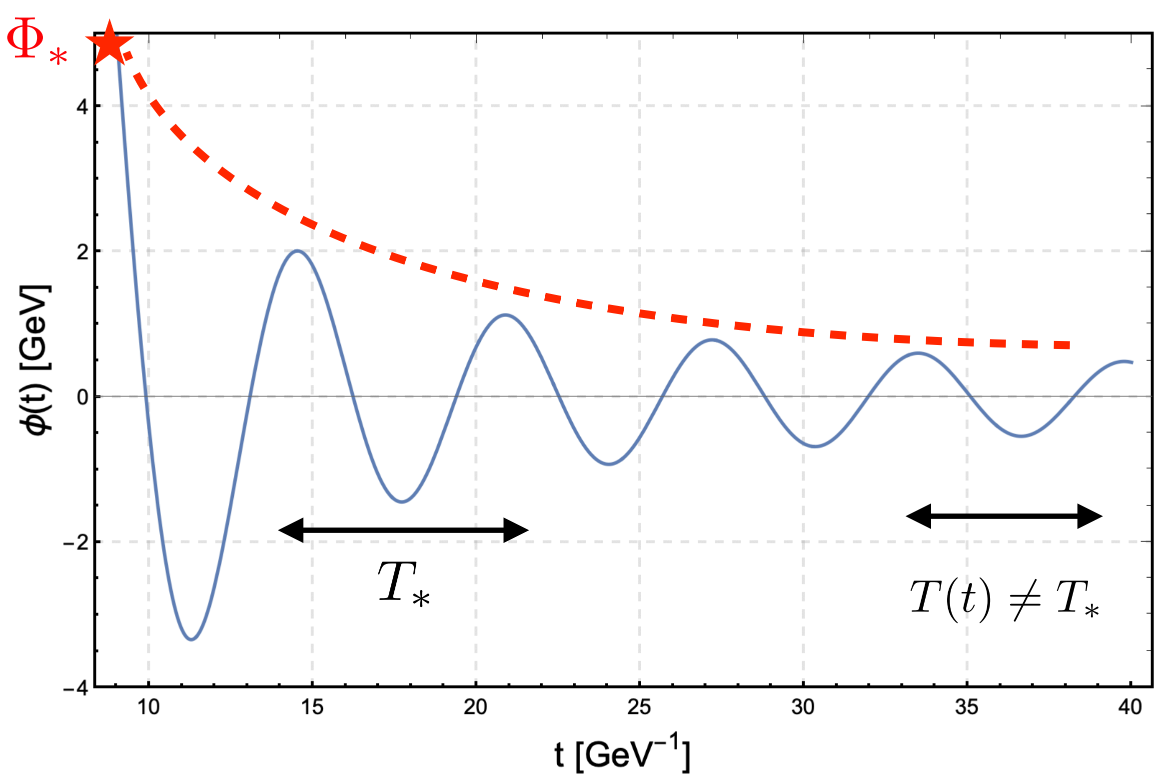
$$\tilde{\phi} = \frac{\phi}{f_*} \quad \tilde{\varphi} = \frac{\varphi}{f_*} \quad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \quad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

Example: $\phi(t) \simeq \Phi_* \times f_{\rm osc}(t)$



Writing a model

➤ Equations solved in (dimensionless) ***program variables***:

Choose:

$$\{\alpha, \omega_*, f_*\}$$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

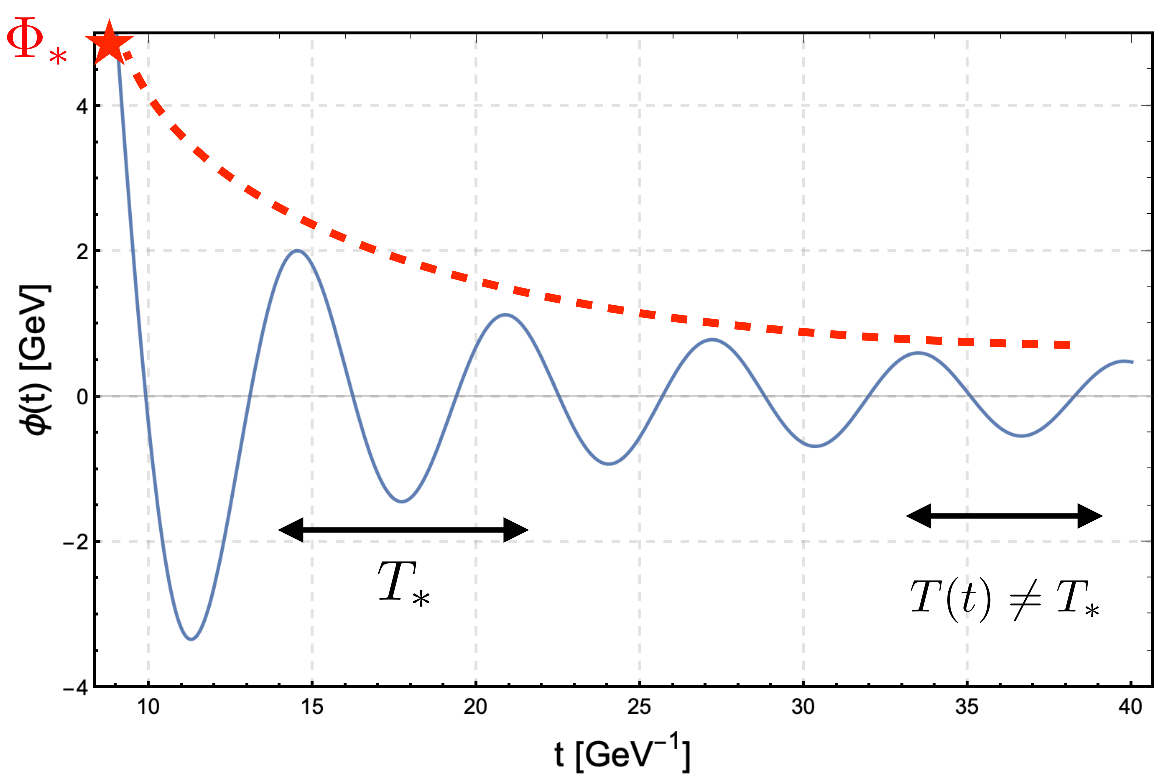
$$\tilde{\phi} = \frac{\phi}{f_*} \qquad \tilde{\varphi} = \frac{\varphi}{f_*} \qquad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \qquad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

Example: $\phi(t) \simeq \Phi_* \times f_{\rm osc}(t)$



$$\begin{cases} f_* = \Phi_* \\ \omega_* = 1/T_* \\ \alpha \longrightarrow \text{Make period constant in } \tilde{\eta} \end{cases}$$

Writing a model

- Equations solved in (dimensionless) ***program variables***:

Choose:

$$\{\alpha, \omega_*, f_*\}$$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

$$\tilde{\phi} = \frac{\phi}{f_*} \quad \tilde{\varphi} = \frac{\varphi}{f_*} \quad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \quad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

- Write **scalar potential** and first and second derivatives in **one file** (*model.h*)

$$\tilde{V}(\tilde{\phi}, |\tilde{\varphi}|, |\widetilde{\Phi}|) \equiv \frac{1}{f_*^2\omega_*^2} V(f_*\tilde{\phi}, f_*|\tilde{\varphi}|, f_*|\widetilde{\Phi}|)$$

$$\frac{\partial \tilde{V}}{\partial \tilde{\phi}}, \frac{\partial \tilde{V}}{\partial |\tilde{\varphi}|}, \frac{\partial \tilde{V}}{\partial |\tilde{\Phi}|}, \frac{\partial^2 \tilde{V}}{\partial \tilde{\phi}^2}, \frac{\partial^2 \tilde{V}}{\partial |\tilde{\varphi}|^2}, \frac{\partial^2 \tilde{V}}{\partial |\tilde{\Phi}|^2},$$

Writing a model

- Equations solved in (dimensionless) ***program variables***:

Choose:

$$\{\alpha, \omega_*, f_*\}$$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt \qquad d\tilde{x}^i \equiv \omega_* dx^i$$

Space and time

$$\tilde{\phi} = \frac{\phi}{f_*} \qquad \tilde{\varphi} = \frac{\varphi}{f_*} \qquad \tilde{\Phi} = \frac{\Phi}{f_*}$$

Scalar fields

$$\widetilde{A}_\mu = \frac{A_\mu}{\omega_*} \qquad \widetilde{B}^a_\mu = \frac{B^a_\mu}{\omega_*}$$

Gauge fields

- Write **scalar potential** and first and second derivatives in **one file** (*model.h*)

$$\tilde{V}(\tilde{\phi}, |\tilde{\varphi}|, |\widetilde{\Phi}|) \equiv \frac{1}{f_*^2\omega_*^2} V(f_*\tilde{\phi}, f_*|\tilde{\varphi}|, f_*|\widetilde{\Phi}|)$$

$$\frac{\partial \tilde{V}}{\partial \tilde{\phi}}, \quad \frac{\partial \tilde{V}}{\partial |\tilde{\varphi}|}, \quad \frac{\partial \tilde{V}}{\partial |\tilde{\Phi}|}, \quad \frac{\partial^2 \tilde{V}}{\partial \tilde{\phi}^2}, \quad \frac{\partial^2 \tilde{V}}{\partial |\tilde{\varphi}|^2}, \quad \frac{\partial^2 \tilde{V}}{\partial |\tilde{\Phi}|^2},$$

- **Parameters** passed via **one file** (input.txt)

(no need to re-compile !)

```
#Output
outputfile = ./

#Evolution
expansion = true
evolver = VV2

#Lattice
N = 32
dt = 0.01
kIR = 0.75
nBinsSpectra = 55

#Times
tOutputFreq = 0.1
tOutputInfreq = 1
tMax = 300

#IC
kCutOff = 1.75
initial_amplitudes = 7.42675e18 0 # homogeneous amplitudes in GeV
initial_momenta = -6.2969e30 0    # homogeneous amplitudes in GeV2

#Model Parameters
lambda = 9e-14
q = 100
```

Self-consistent Expansion

➤ Algorithms use **second Friedmann equation** to **evolve the scale factor**.

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3m_p^2}\langle(\alpha-2)(K_\phi+K_\varphi+K_\Phi)+\alpha(G_\phi+G_\varphi+G_\Phi)+(\alpha+1)V + (\alpha-1)(K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)})\rangle$$

$\langle\ldots\rangle$ represents volume averaging

$$\begin{aligned} K_\phi &= \tfrac{1}{2a^{2\alpha}}\phi'^2 \\ K_\varphi &= \tfrac{1}{a^{2\alpha}}(D_0^A\varphi)^*(D_0^A\varphi) \\ K_\Phi &= \tfrac{1}{a^{2\alpha}}(D_0\Phi)^\dagger(D_0\Phi) \end{aligned} \quad ; \quad \begin{aligned} G_\phi &= \tfrac{1}{2a^2}\textstyle\sum_i(\partial_i\phi)^2 \\ G_\varphi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i^A\varphi)^*(D_i^A\varphi) \\ G_\Phi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i\Phi)^\dagger(D_i\Phi) \end{aligned} \quad ; \quad \begin{aligned} K_{U(1)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_i F_{0i}^2 \\ K_{SU(2)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_{a,i}(G_{0i}^a)^2 \\ G_{U(1)} &= \tfrac{1}{2a^4}\textstyle\sum_{i,j<i}F_{ij}^2 \\ G_{SU(2)} &= \tfrac{1}{2a^4}\textstyle\sum_{a,i,j<i}(G_{ij}^a)^2 \end{aligned}$$

(Kinetic-Scalar) (Gradient-Scalar) (Electric & Magnetic)

Self-consistent Expansion

- Algorithms use **second Friedmann equation** to **evolve the scale factor**.
- The **first Friedmann equation** is used to check the accuracy of the simulation.

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3m_p^2}\langle(\alpha-2)(K_\phi+K_\varphi+K_\Phi)+\alpha(G_\phi+G_\varphi+G_\Phi)+(\alpha+1)V + (\alpha-1)(K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)})\rangle$$

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3m_p^2}\langle K_\phi+K_\varphi+K_\Phi+G_\phi+G_\varphi+G_\Phi+K_{U(1)}+G_{U(1)}+K_{SU(2)}+G_{SU(2)}+V\rangle$$

$\langle\ldots\rangle$ represents volume averaging

$$\begin{aligned} K_\phi &= \tfrac{1}{2a^{2\alpha}}\phi'^2 \\ K_\varphi &= \tfrac{1}{a^{2\alpha}}(D_0^A\varphi)^*(D_0^A\varphi) \\ K_\Phi &= \tfrac{1}{a^{2\alpha}}(D_0\Phi)^\dagger(D_0\Phi) \end{aligned} \quad ; \quad \begin{aligned} G_\phi &= \tfrac{1}{2a^2}\textstyle\sum_i(\partial_i\phi)^2 \\ G_\varphi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i^A\varphi)^*(D_i^A\varphi) \\ G_\Phi &= \tfrac{1}{a^2}\textstyle\sum_i(D_i\Phi)^\dagger(D_i\Phi) \end{aligned} \quad ; \quad \begin{aligned} K_{U(1)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_i F_{0i}^2 \\ K_{SU(2)} &= \tfrac{1}{2a^{2+2\alpha}}\textstyle\sum_{a,i}(G_{0i}^a)^2 \\ G_{U(1)} &= \tfrac{1}{2a^4}\textstyle\sum_{i,j<i}F_{ij}^2 \\ G_{SU(2)} &= \tfrac{1}{2a^4}\textstyle\sum_{a,i,j<i}(G_{ij}^a)^2 \end{aligned}$$

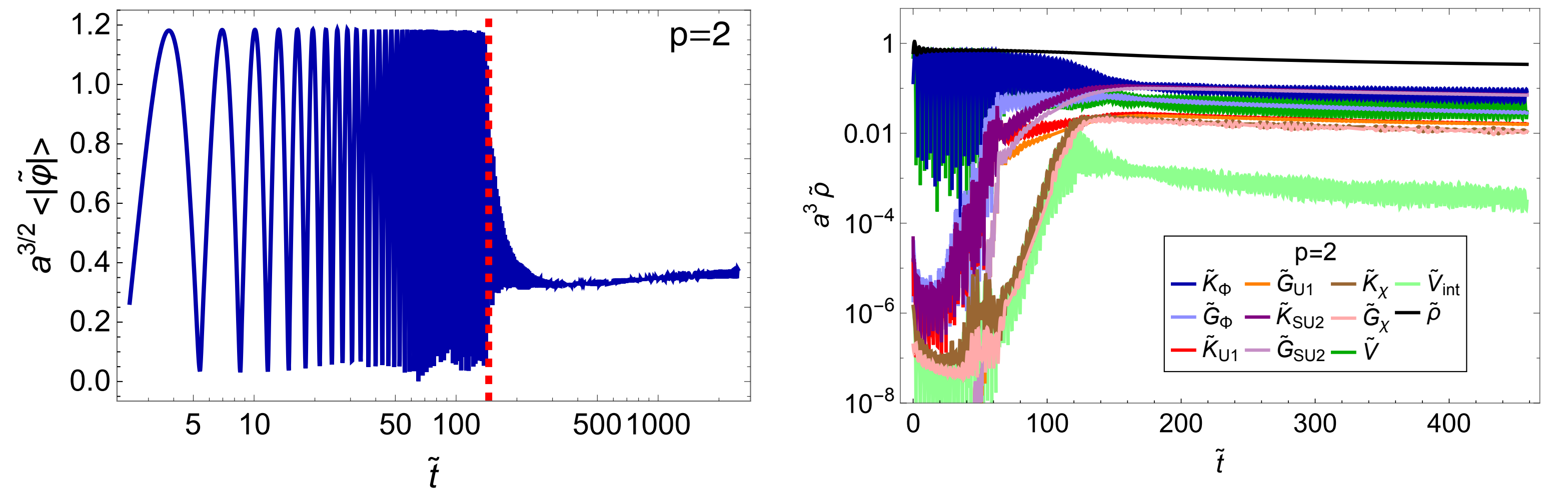
(Kinetic-Scalar) (Gradient-Scalar) (Electric & Magnetic)

Transparencies worked out together with Paco Torrentí

Output from your Run

Output Types

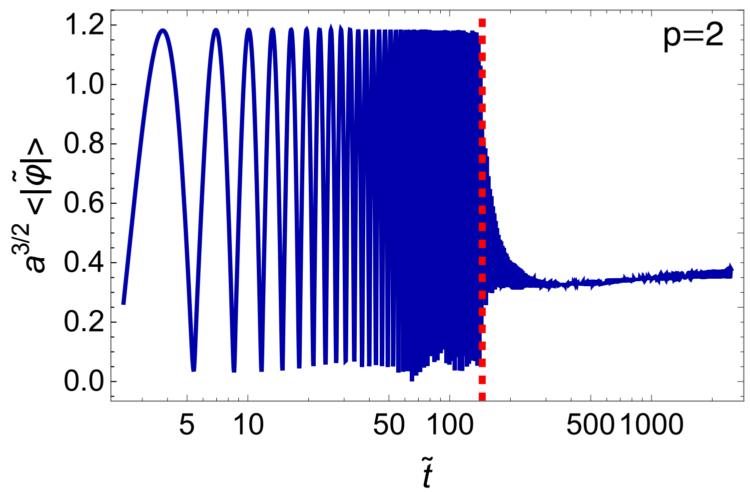
Volume averages: variance, energies, etc

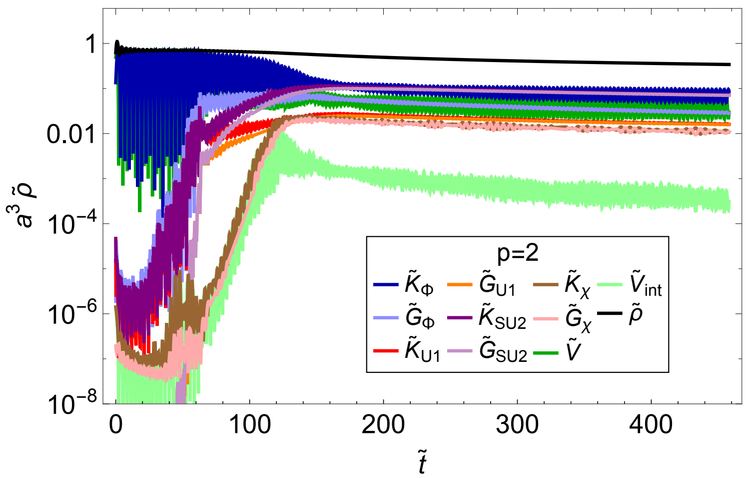


Output from your Run

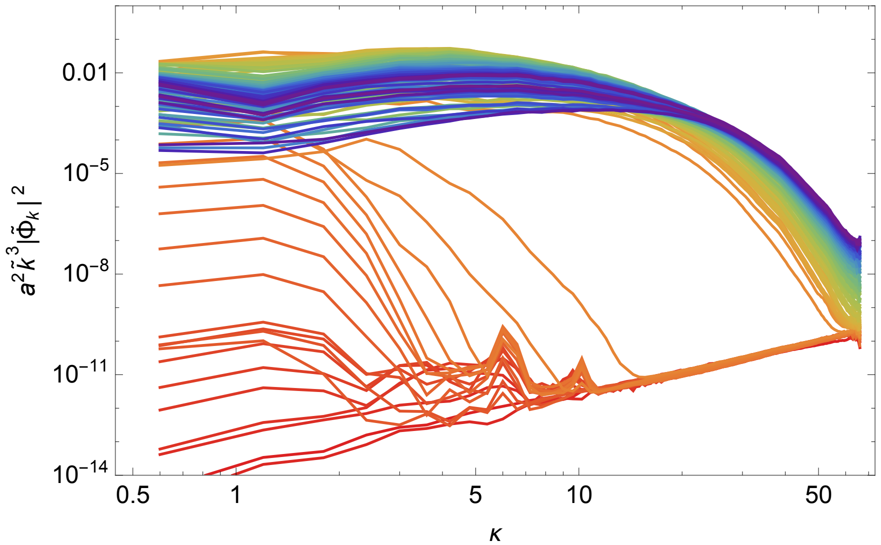
Output Types

Volume averages: variance, energies, etc





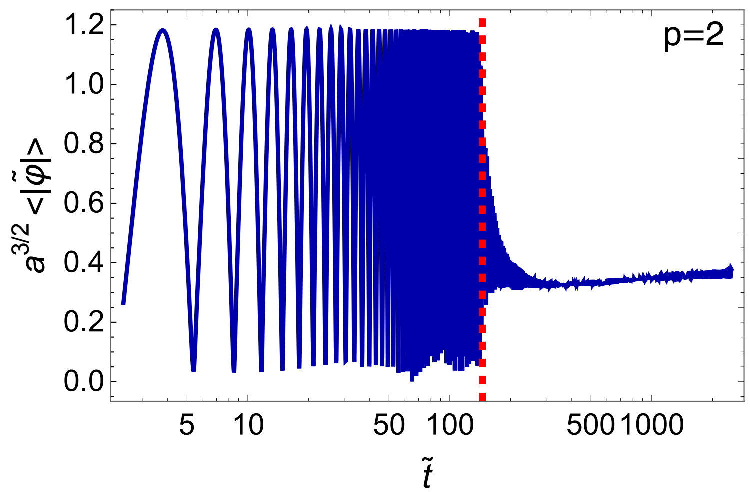
Fld Spectra: Raw/Binned

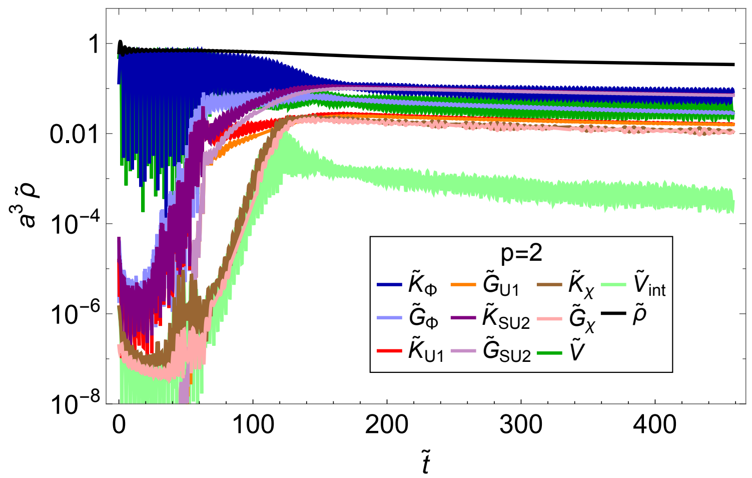


Output from your Run

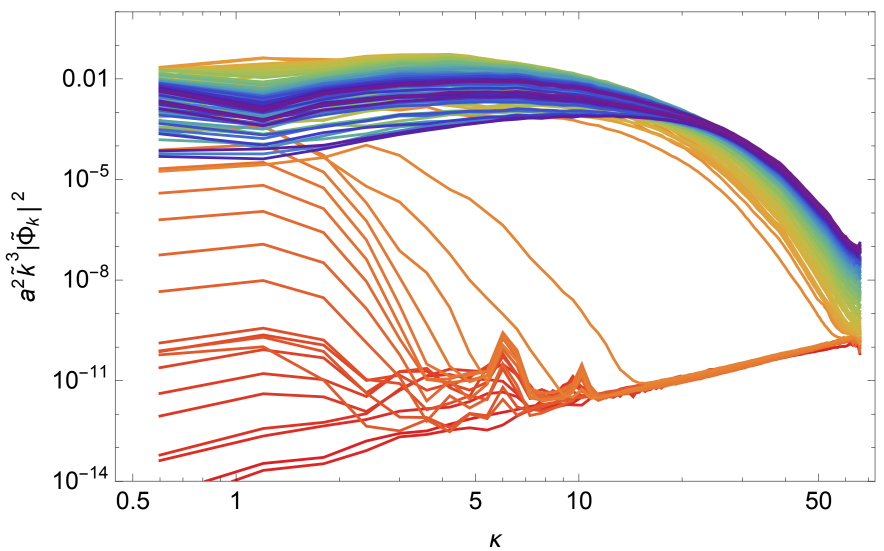
Output Types

Volume averages: variance, energies, etc

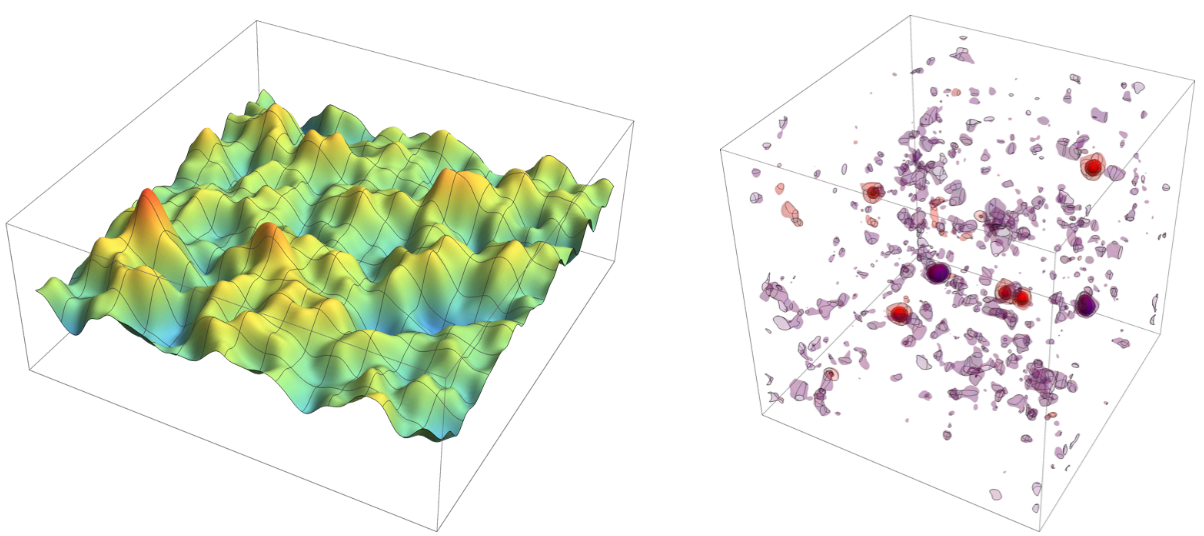




Fld Spectra: Raw/Binned



Snapshots: 2D/3D distribution



CosmoLattice

http://www.cosmolattice.net/

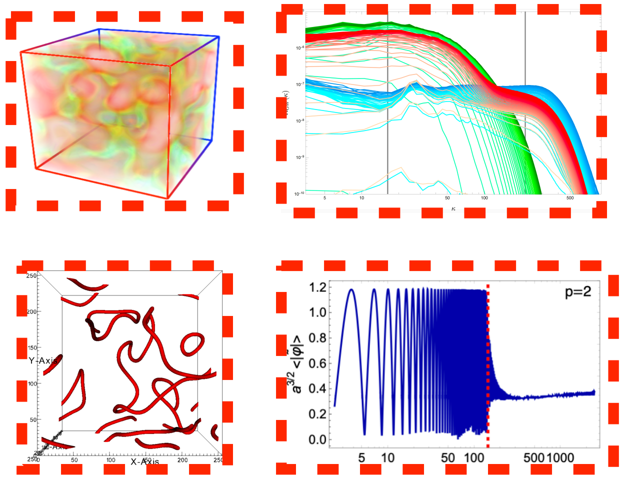
CosmoLattice

Physical Problem

* Init Conditions
* Eqs. of Motion

* Choose Lattice: dt, N, dx
* Choose Algorithm $\mathcal{O}(dt^n)$
* Choose Param: g, m, ...
* Choose Observables

Output



CosmoLattice

http://www.cosmolattice.net/

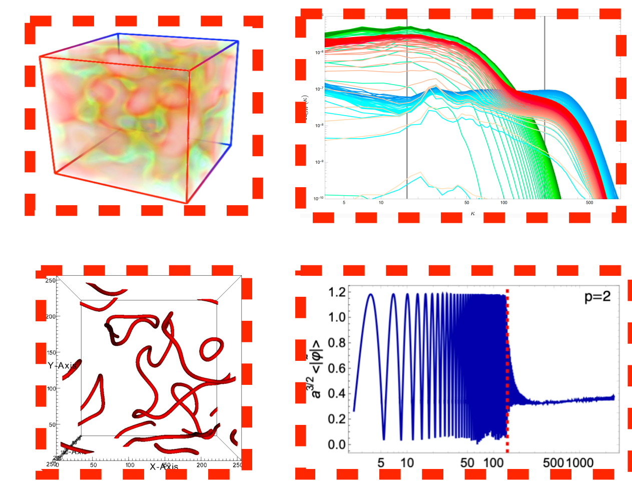
Physical Problem

* Init Conditions
* Eqs. of Motion

CosmoLattice

* New Physical Problem
* Choose Lattice: dt, N, dx
* Choose Algorithm $\mathcal{O}(dt^n)$
* Choose Param: g, m, ...
* Choose Observables

Output



CosmoLattice

http://www.cosmolattice.net/

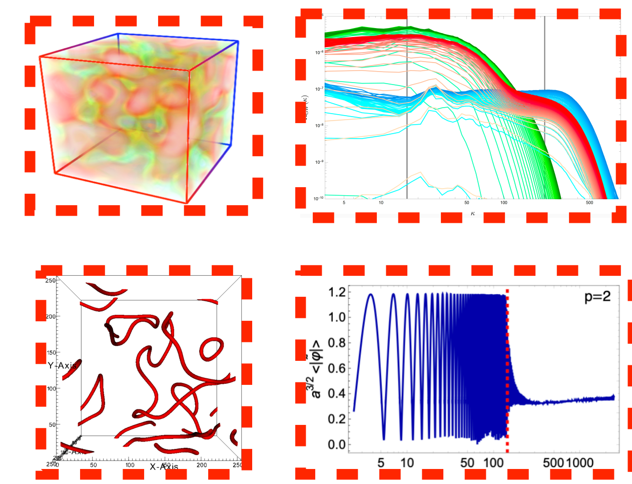
CosmoLattice

Physical Problem

* Init Conditions
* Eqs. of Motion

* New Physical Problem
* Choose Lattice: dt, N, dx
* Choose Algorithm $\mathcal{O}(dt^n)$
* Choose Param: g, m, ...
* Choose Observables

Output



CL is a platform for field theories

You choose the problem to solve !

CosmoLattice

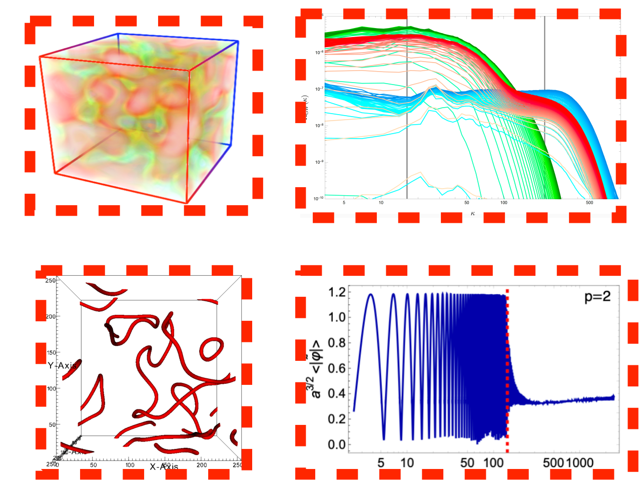
http://www.cosmolattice.net/

Physical Problem

* Init Conditions
* Eqs. of Motion

CosmoLattice

Output



Basic use of CL:
Black Box

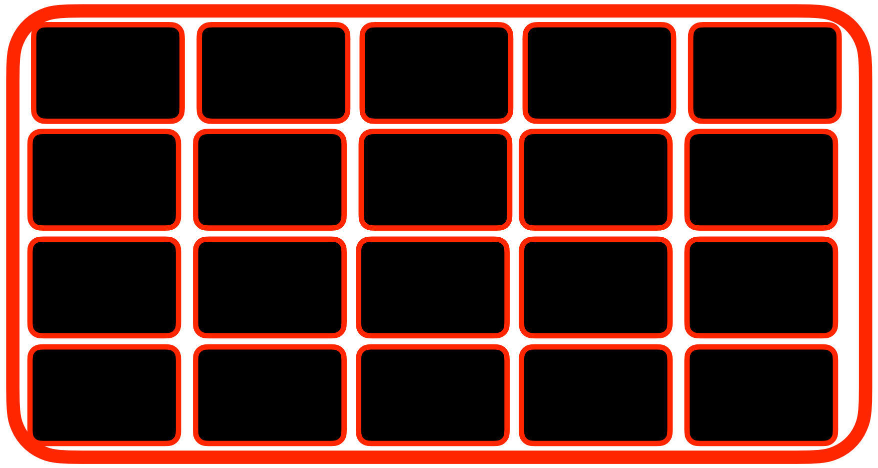
CosmoLattice

http://www.cosmolattice.net/

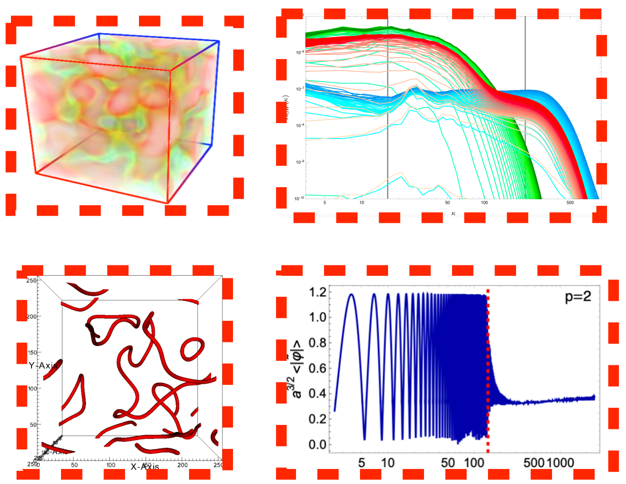
Physical Problem

* Init Conditions
* Eqs. of Motion

CosmoLattice



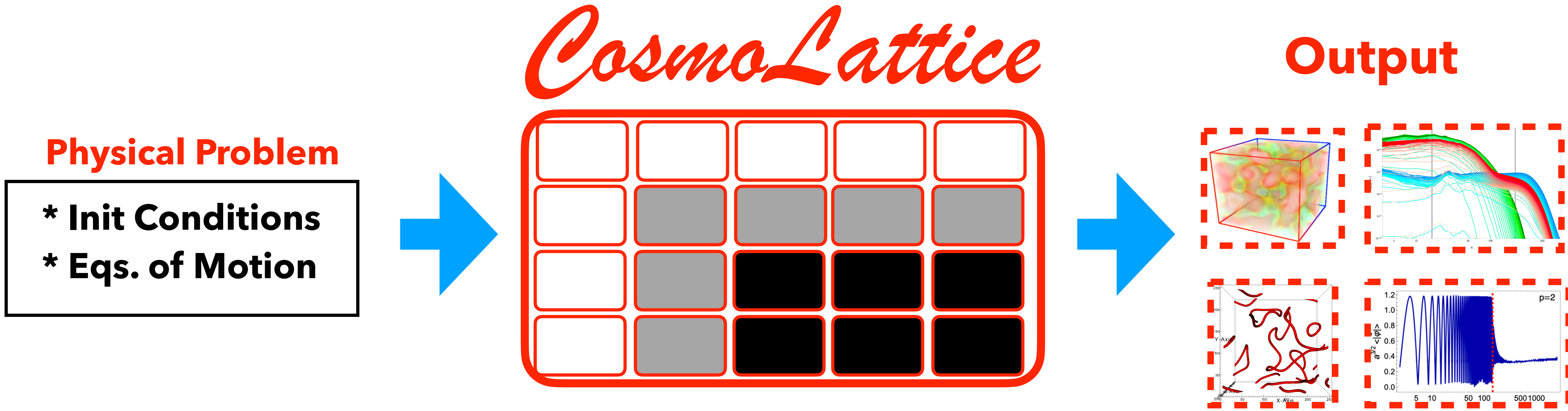
Output



Basic use of CL:
Set of Black Boxes

CosmoLattice

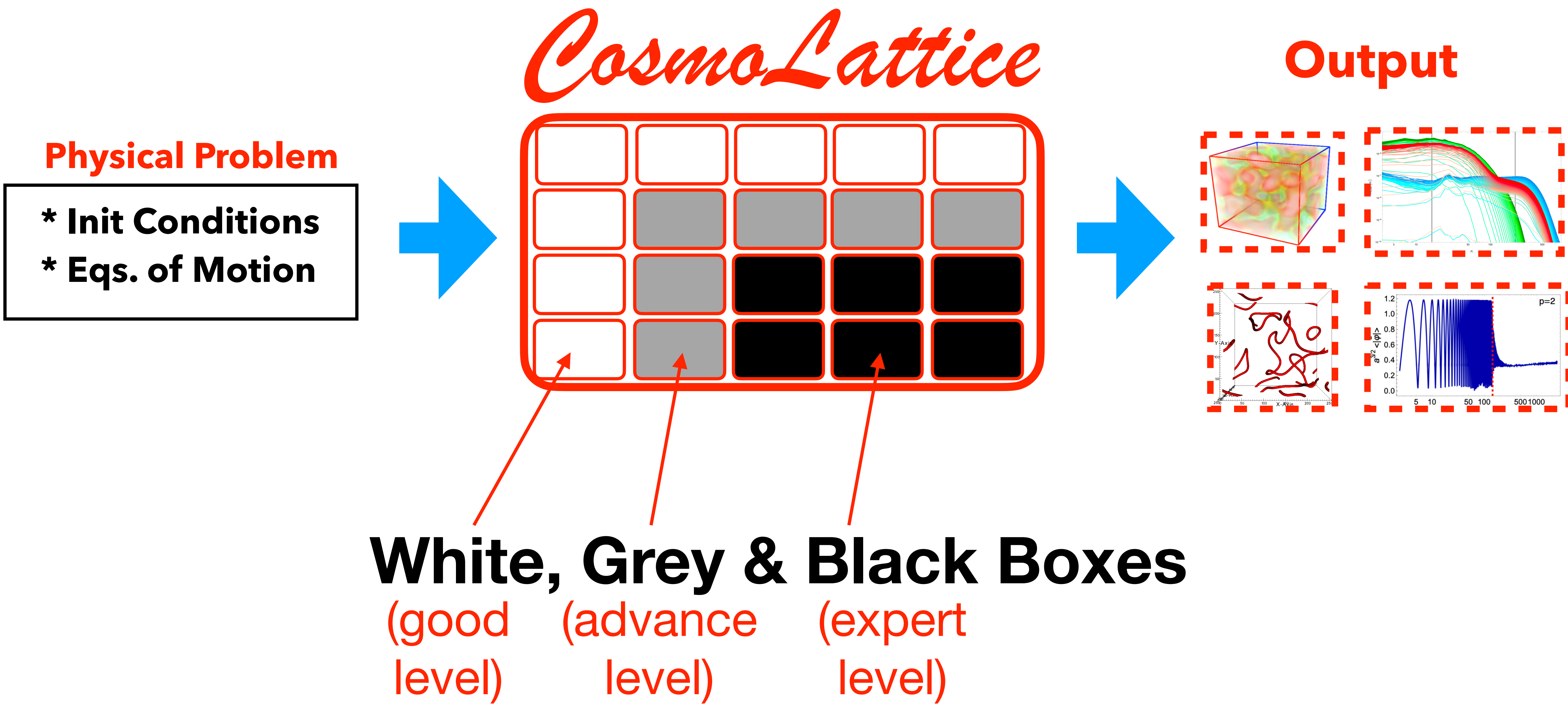
http://www.cosmolattice.net/



Proper use of CL:
White, Grey & Black Boxes

CosmoLattice

http://www.cosmolattice.net/



CosmoLattice

http://www.cosmolattice.net/

- **CL so far (v1.0, Public):**
  - Global scalar field dynamics
  - U(1) scalar-gauge dynamics
  - SU(2) scalar-gauge dynamics

Ready to use !

CosmoLattice

http://www.cosmolattice.net/

- **CL so far (v1.0, Public):**
  - Global scalar field dynamics
  - U(1) scalar-gauge dynamics
  - SU(2) scalar-gauge dynamics

Main lectures

Ready to use !

Daniel G. Figueroa
IFIC UV/CSIC, Spain

Adrien Florio
Stony Brook U., USA

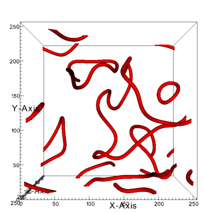
Francisco Torrenti
U. Basel, Switzerland

CosmoLattice

http://www.cosmolattice.net/

- **CL so far (v1.0, Public):**
  - Global scalar field dynamics
  - U(1) scalar-gauge dynamics
  - SU(2) scalar-gauge dynamics

  } **Ready to use !**

- **CL update (v2.0, to be released by ~2023):**
  - ✓ Gravitational waves $\Box h_{ij} = 2\Pi^{\rm TT}_{ij}$
  - ✓ Axion-like couplings $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$
  - ✓ Non-minimal coupling $\xi\phi^2 R$
  - ✓ Cosmic String Networks 
  - …

  } **Implemented, but not (yet) released**

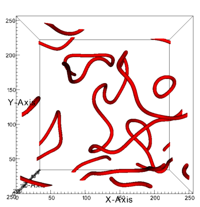
CosmoLattice

http://www.cosmolattice.net/

- **CL so far (v1.0, Public):**
  - Global scalar field dynamics
  - U(1) scalar-gauge dynamics
  - SU(2) scalar-gauge dynamics

Ready to use !

- **CL update (v2.0, to be released by ~2023):**
  - ✓ Gravitational waves $\Box h_{ij} = 2\Pi_{ij}^{\rm TT}$
  - ✓ Axion-like couplings $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$
  - ✓ Non-minimal coupling $\xi\phi^2 R$
  - ✓ Cosmic String Networks



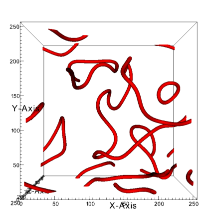
  - ...

Implemented, but not (yet) released

CL v1.1: Just released in May 2022 !

CosmoLattice

http://www.cosmolattice.net/

- **CL update (v2.0, to be released by ~2023):**
  - ✔ Gravitational waves $\Box h_{ij} = 2\Pi^{TT}_{ij}$
  - ✔ Axion-like couplings $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$
  - ✔ Non-minimal coupling $\xi\phi^2 R$
  - ✔ Cosmic String Networks 
  - ...

Topical lectures

Implemented, but not (yet) released

CosmoLattice

http://www.cosmolattice.net/

Joanes Lizarraga
(UPV/EHU, Bilbao, Spain)

Ben STEFANEK
(Zurich Univ., Switzerland)

Toby OPFERKUCH
(Berkeley Univ., USA)

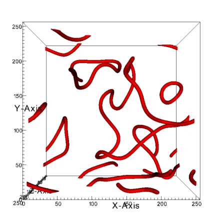
Jorge BAEZA-BALLESTEROS
(IFIC, Valencia, Spain)

Nicolás LOAYZA
(IFIC, Valencia, Spain)

Ken MARSCHALL
(Basel Univ., Switzerland)

Ander URIO
(UPV/EHU, Bilbao, Spain)

- **CL update (v2.0, to be released by ~2023):**
  - ✓ Gravitational waves $\Box h_{ij} = 2\Pi_{ij}^{TT}$
  - ✓ Axion-like couplings $\phi F_{\mu\nu}\tilde{F}^{\mu\nu}$
  - ✓ Non-minimal coupling $\xi\phi^2 R$
  - ✓ Cosmic String Networks



  - …

Implemented, but not (yet) released

Topical lectures

— Lecture 1 —

Welcome to the Lattice

* L1.a: Overview of CosmoLattice (CL) ✓
* L1.b: What is really a Lattice ?

— Lecture 1 —

Welcome to the Lattice

* L1.a: Overview of CosmoLattice (CL) ✓
* L1.b: What is really a Lattice ?

5 min
Break ?

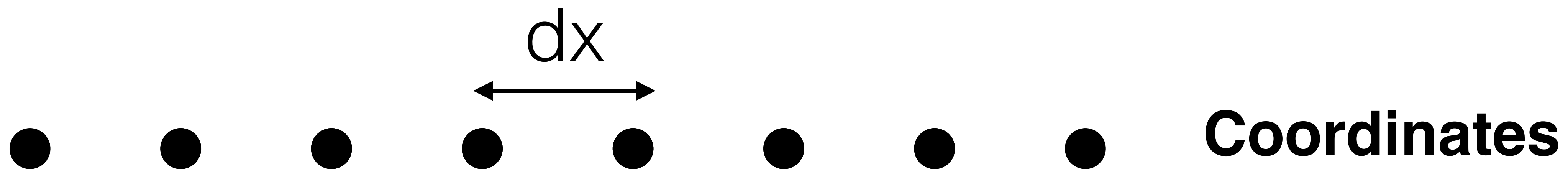
— Lecture 1.b —

What is really a Lattice ?

Primer on Lattice Techniques

1D Lattice

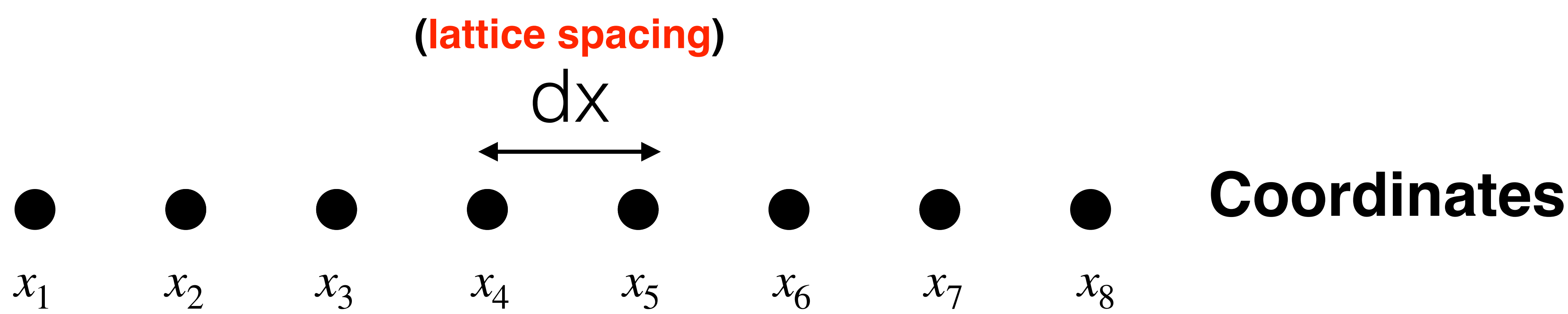
(Set of regular discrete coordinates)



Primer on Lattice Techniques

1D Lattice

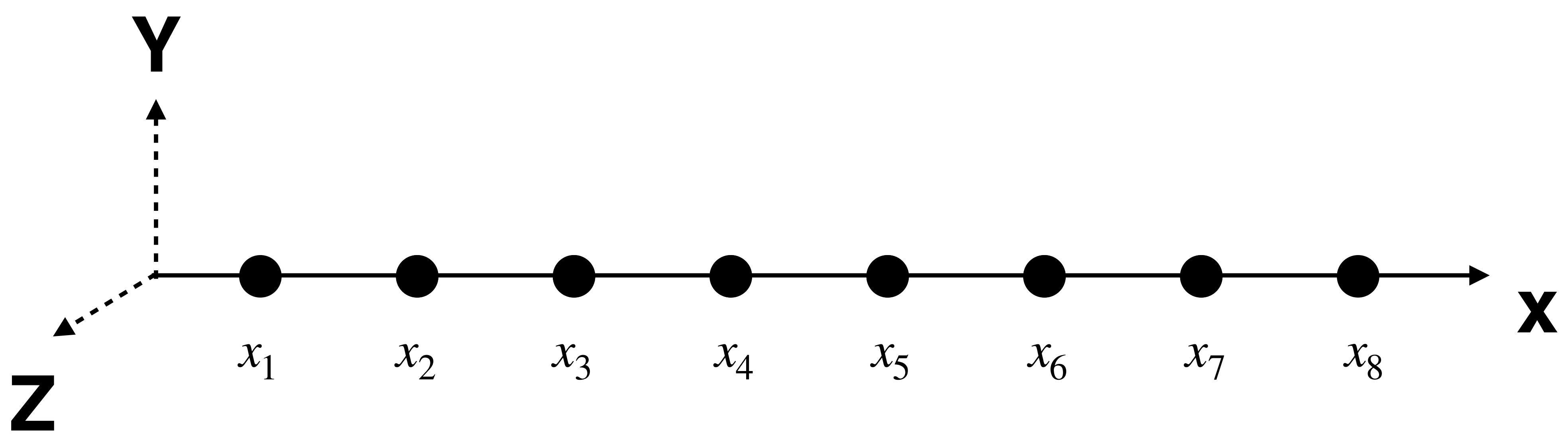
(Set of regular discrete coordinates)



Primer on Lattice Techniques

1D Lattice

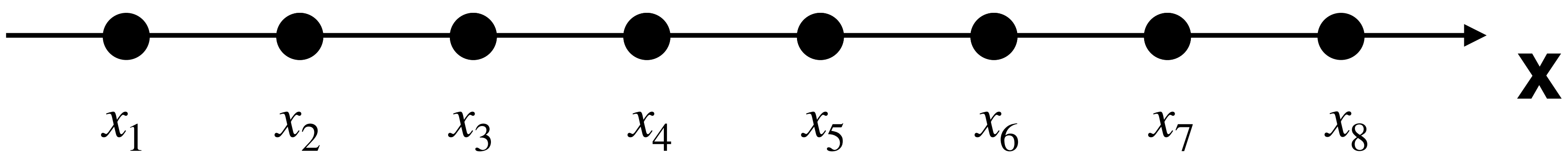
(Set of regular discrete coordinates)



$$\{x_n\}\,,\ n = 1,\ 2,\ \ldots,\ N$$

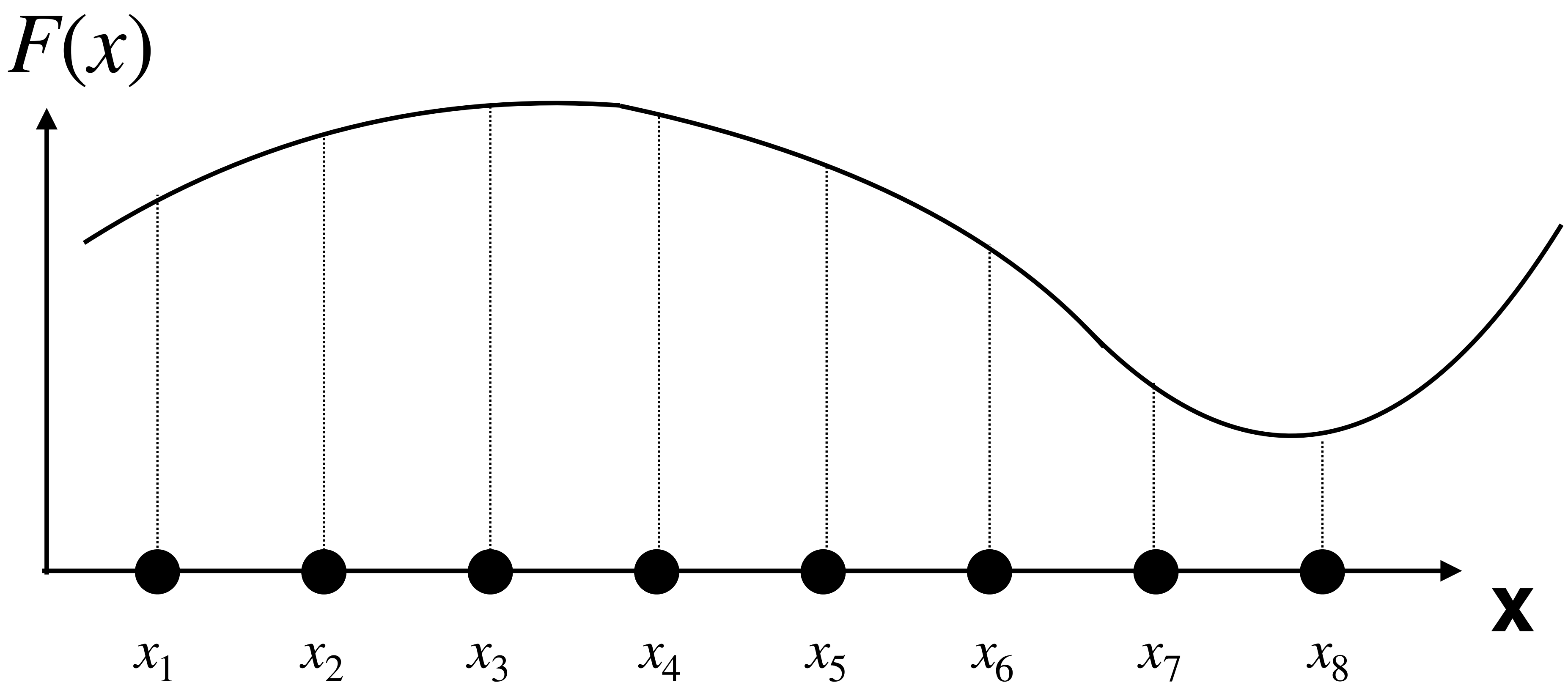
1D Lattice

(Set of regular discrete coordinates)



$$\{x_n\}\,,\ n = 1,\ 2,\ \ldots.,\ N$$

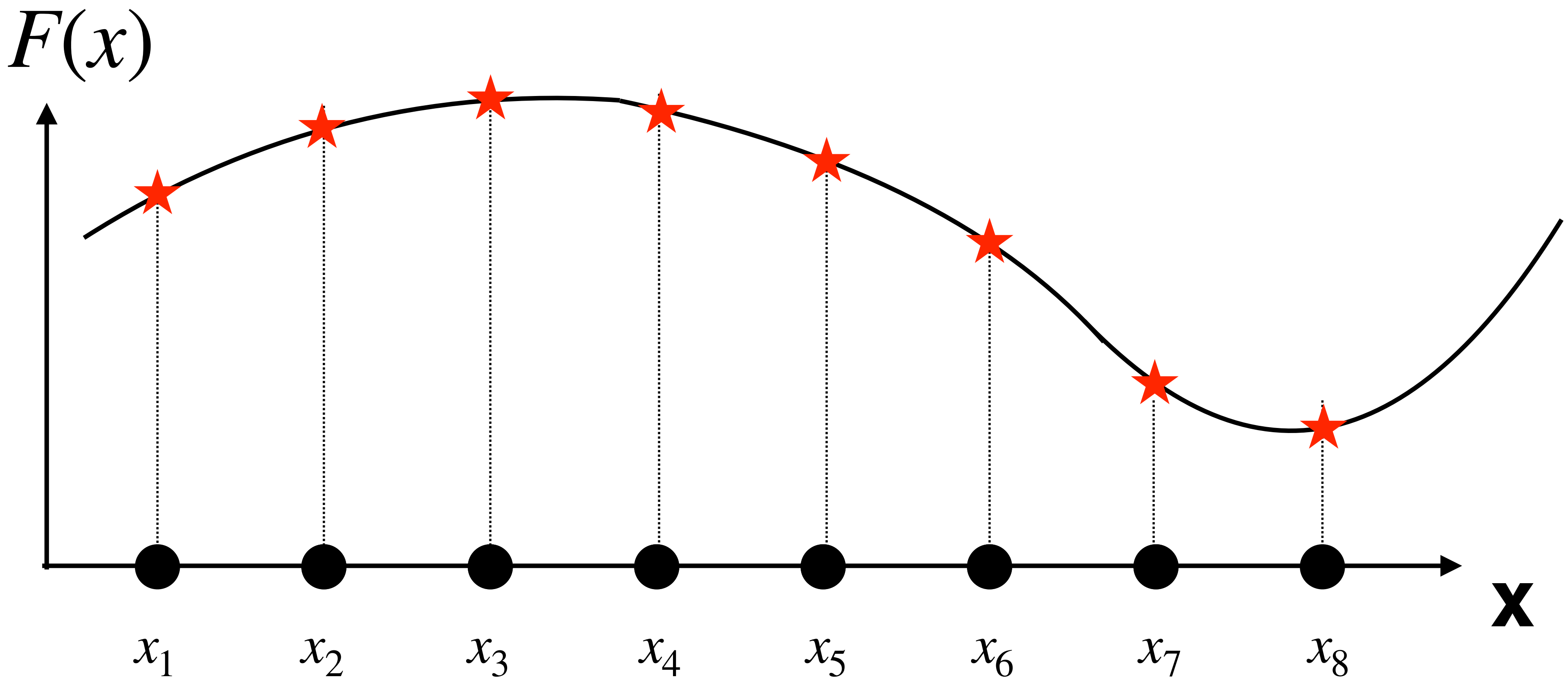
Primer on Lattice Techniques



$$\{x_n\}\ ,\ n = 1,\ 2,\ \ldots.,\ N$$

Primer on Lattice Techniques

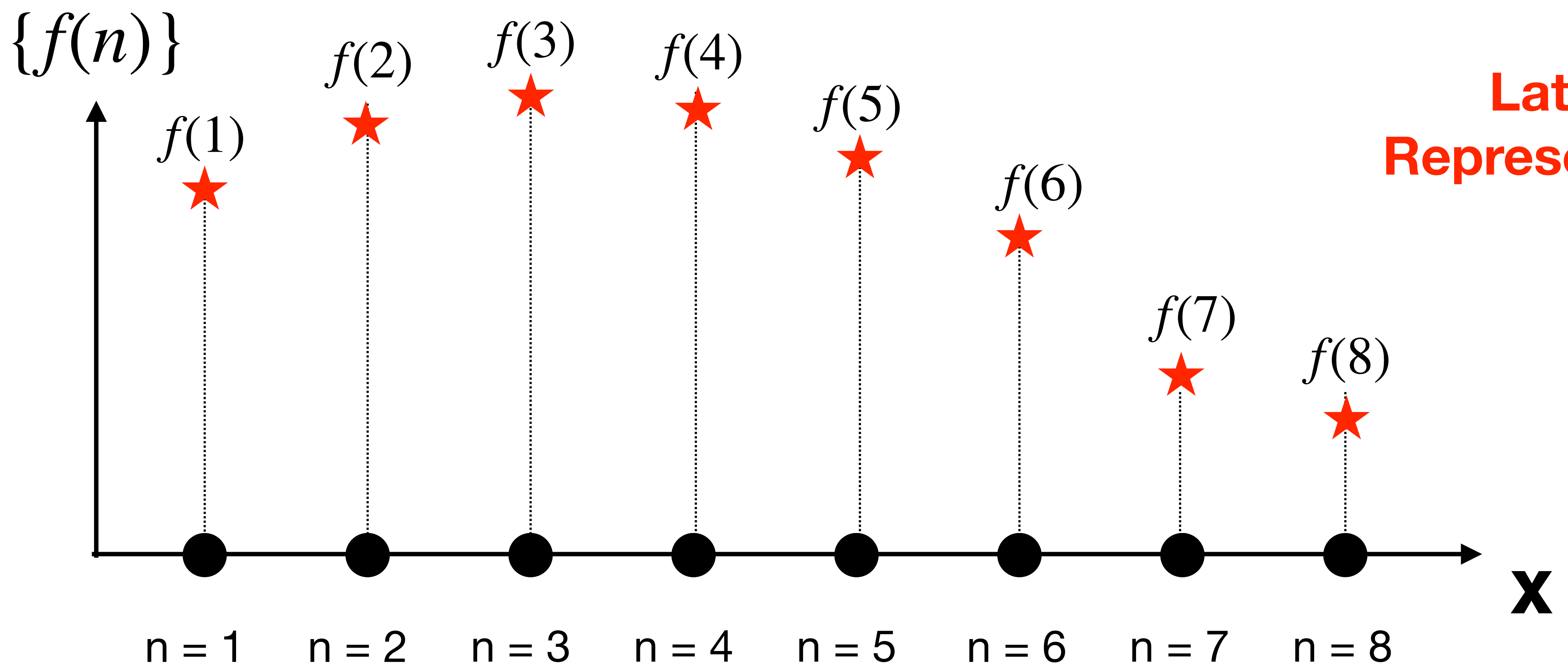
$$f(n) \equiv F(x_n \equiv x_* + ndx)$$



$$\{x_n\}\,,\ n = 1,\ 2,\ \ldots.,\ N$$

Primer on Lattice Techniques

$$f(n) \equiv F(x_n \equiv x_* + ndx)$$

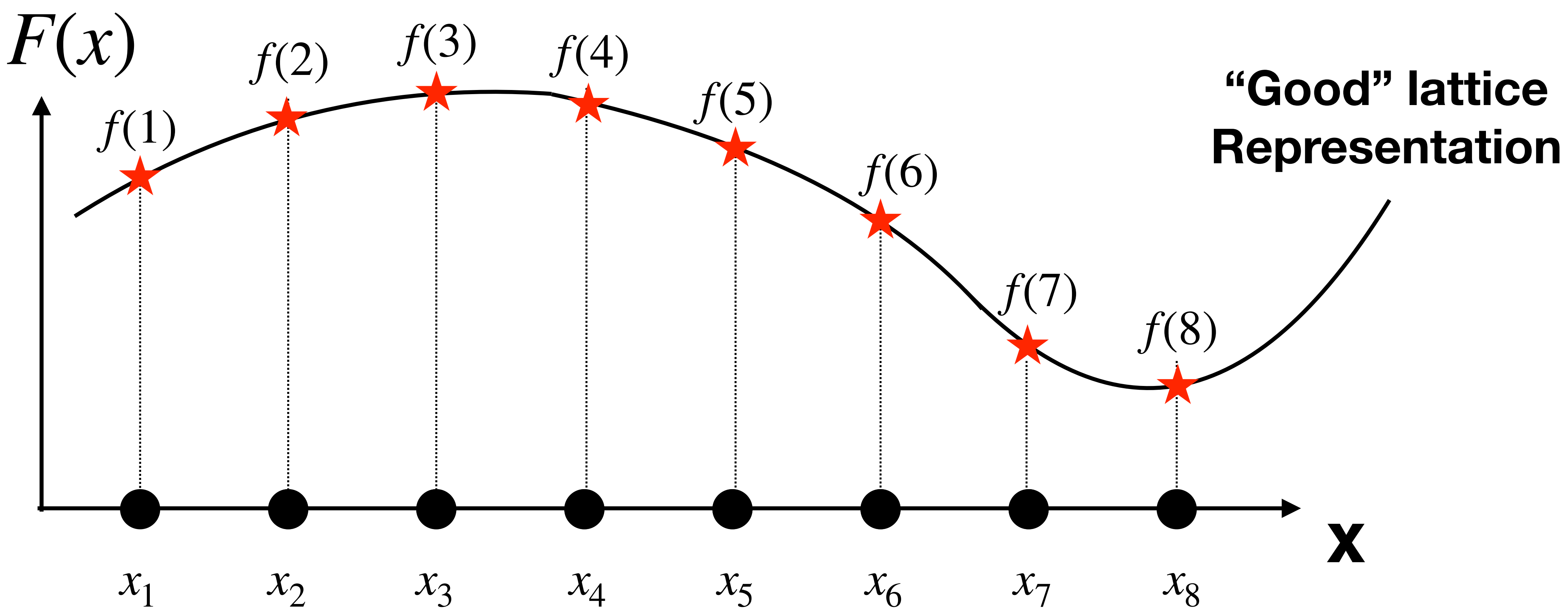


Lattice Representation

$$\{f(n) \equiv F(x_n)\}\,,\; n = 1,\, 2,\, \ldots.,\, N$$

Primer on Lattice Techniques

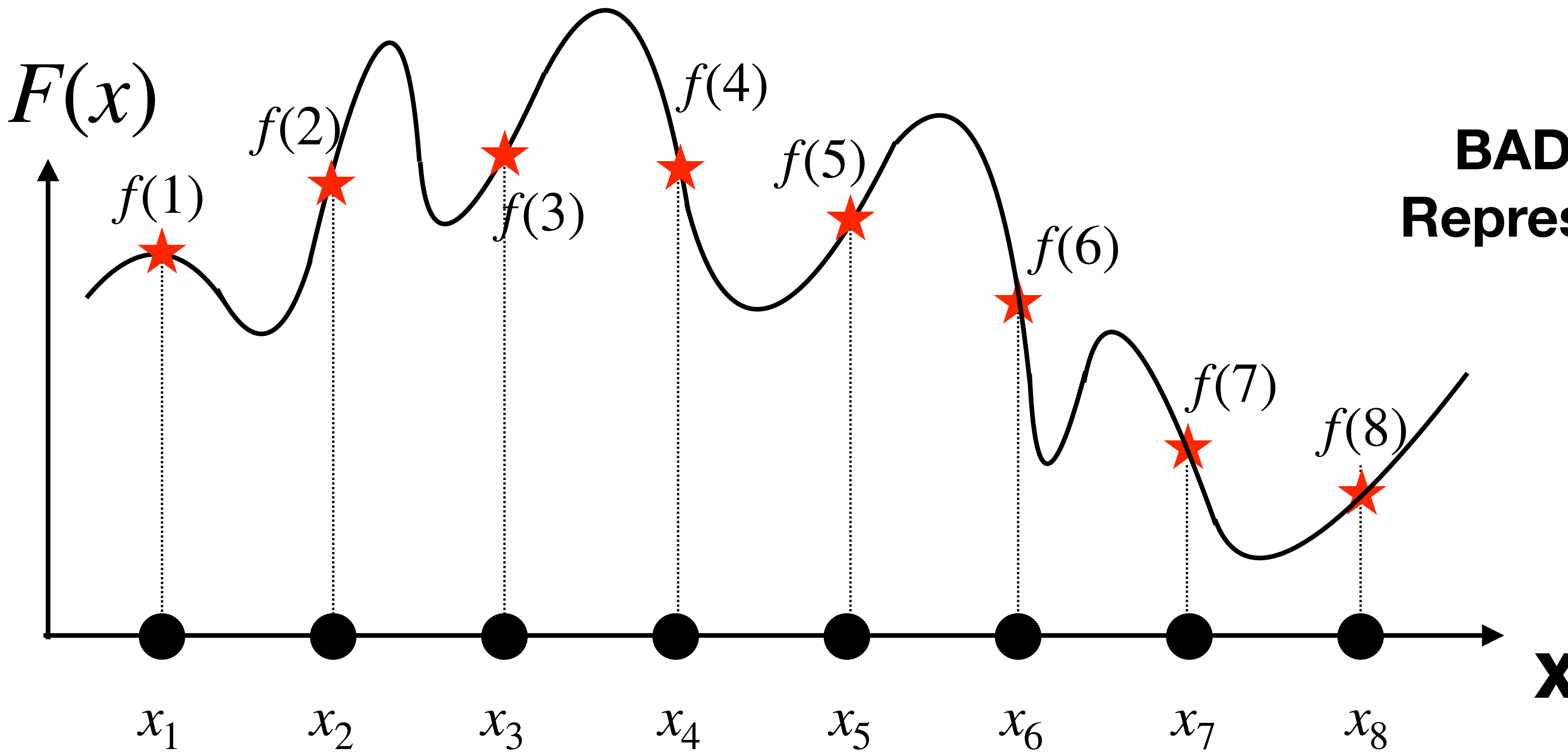
$$f(n) \equiv F(x_n \equiv x_* + n dx)$$



$$\{f(n) \equiv F(x_n)\}\,,\; n = 1,\, 2,\, \ldots.,\, N$$

Primer on Lattice Techniques

$$f(n) \equiv F(x_n \equiv x_* + ndx)$$

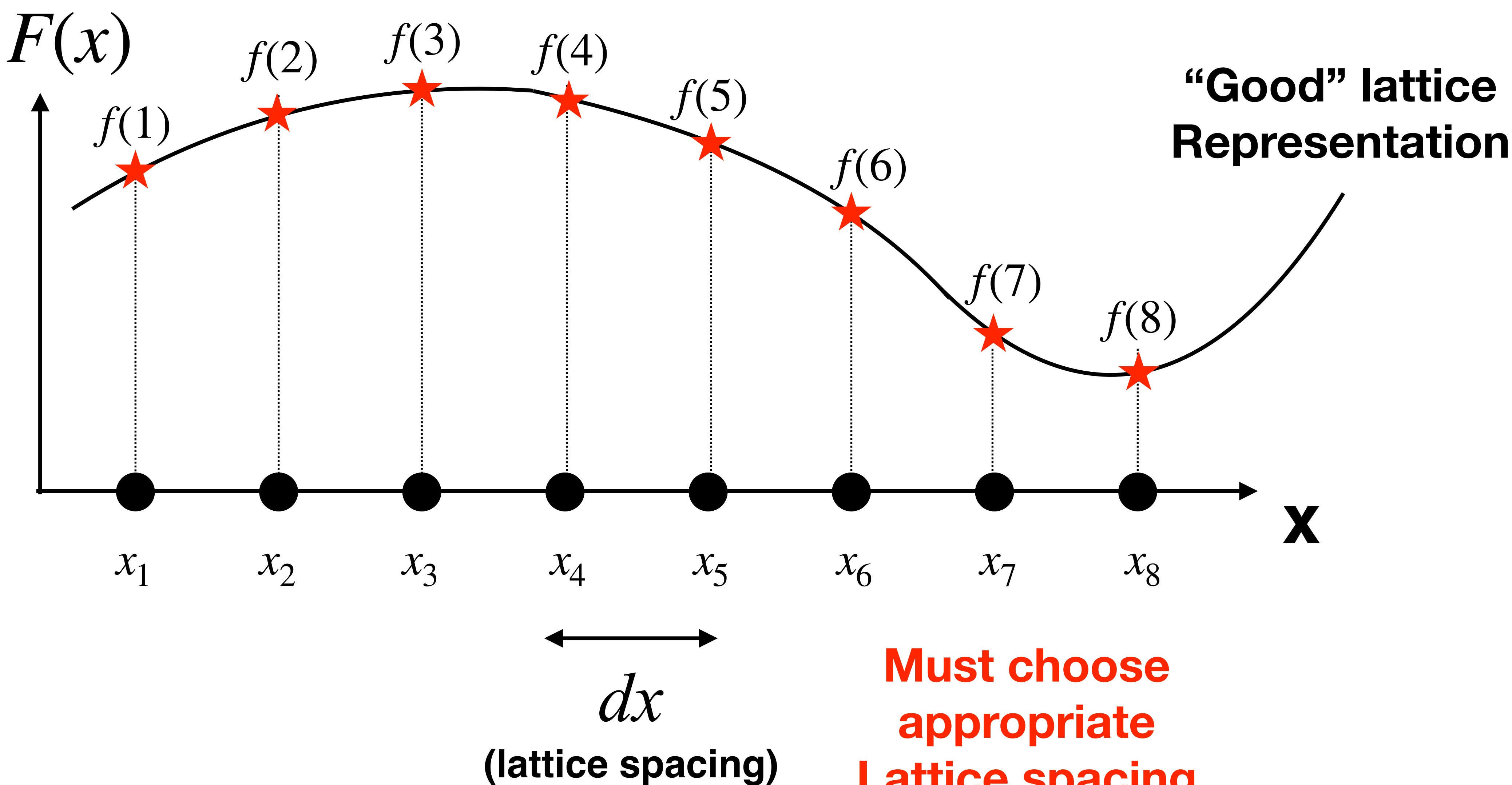


BAD lattice Representation

$$\{f(n) \equiv F(x_n)\}\ ,\ n = 1,\ 2,\ \ldots\ldots,\ N$$

Primer on Lattice Techniques

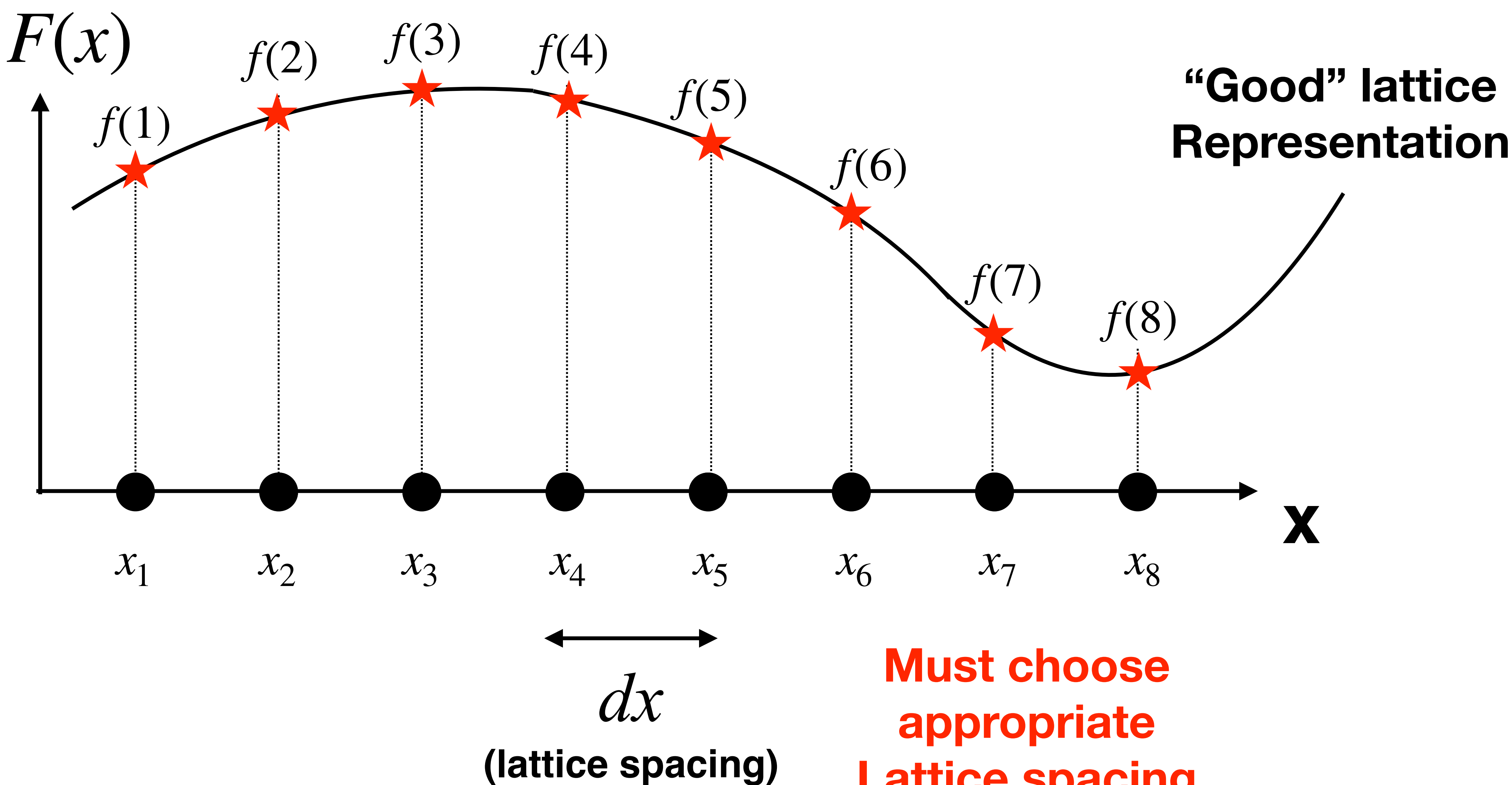
$$f(n) \equiv F(x_n \equiv x_* + ndx)$$



Must choose appropriate Lattice spacing

Primer on Lattice Techniques

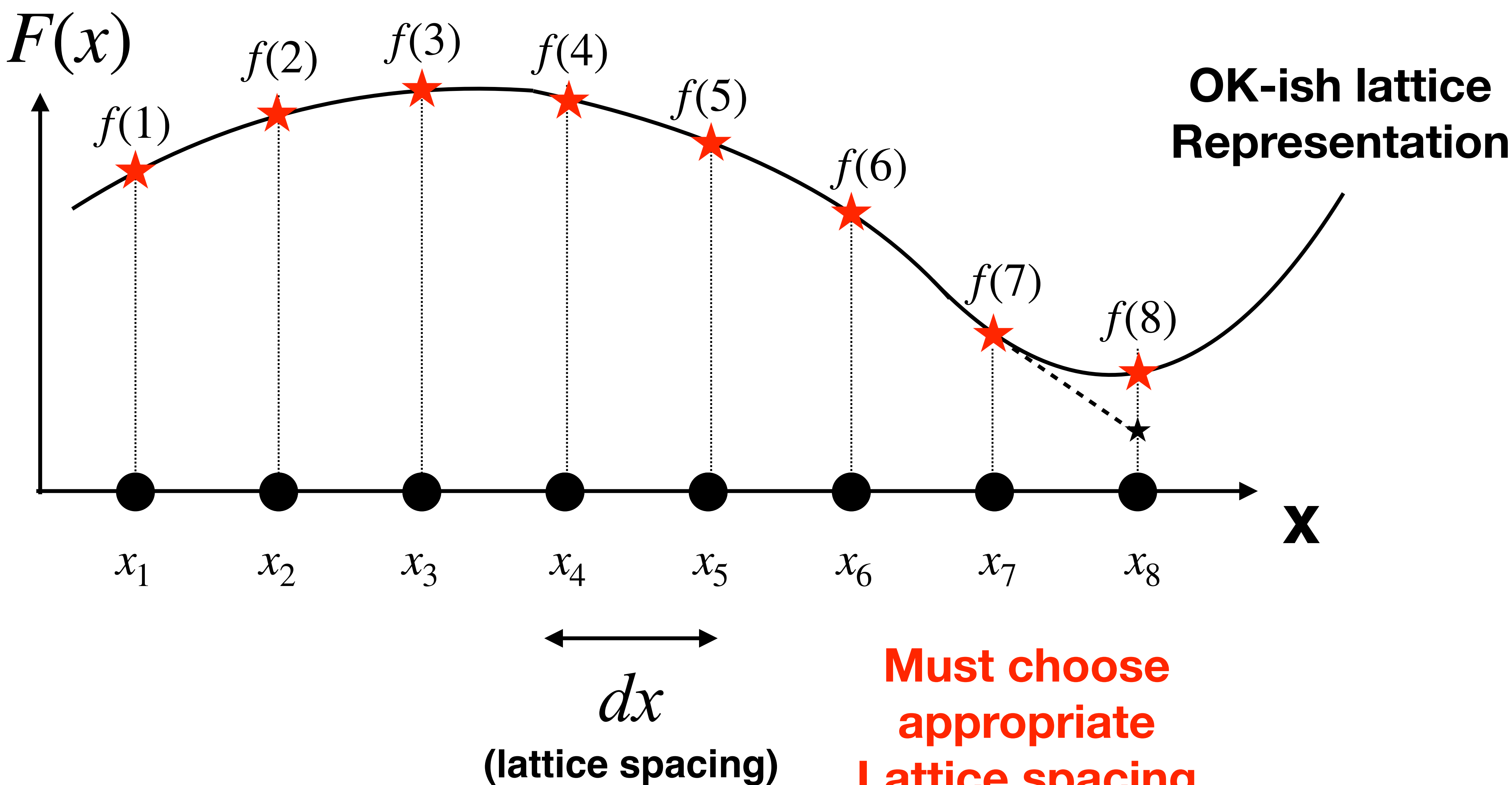
$$f(n) \equiv F(x_n \equiv x_* + ndx)$$



"Good" lattice spacing: $|f(n+1) - f(n)| \sim |F'(x_n)|\, dx$

Primer on Lattice Techniques

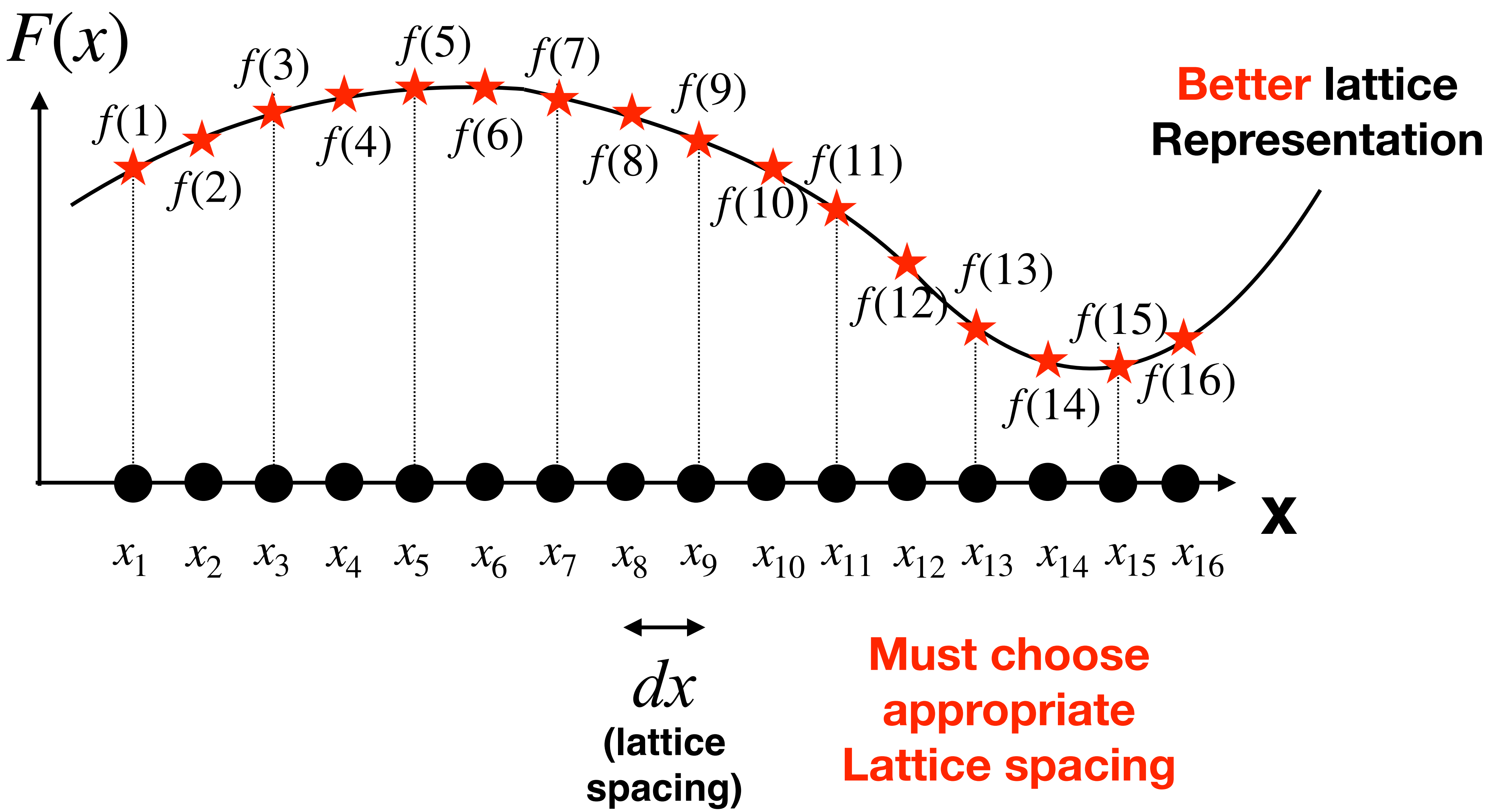
$$f(n) \equiv F(x_n \equiv x_* + n dx)$$



"Good" lattice spacing: $|f(n+1) - f(n)| \sim |F'(x_n)|\, dx$

Primer on Lattice Techniques

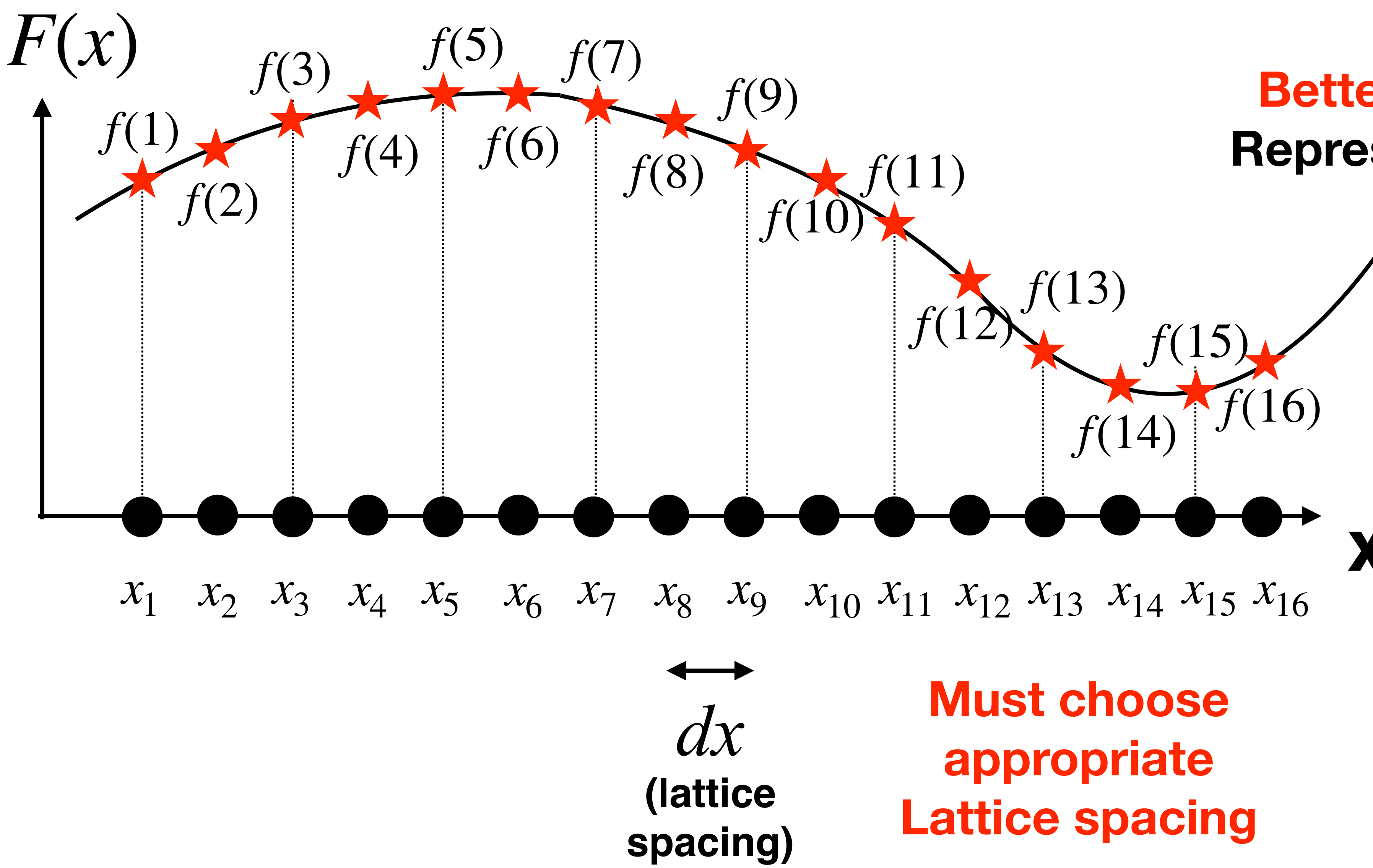
$$f(n) \equiv F(x_n \equiv x_* + n dx)$$



"Good" lattice spacing: $|f(n+1) - f(n)| \sim |F'(x_n)| dx$

Primer on Lattice Techniques

$$f(n) \equiv F(x_n \equiv x_* + ndx)$$



Better lattice Representation

Price to pay
Must store twice as many numbers

Must choose appropriate Lattice spacing

"Good" lattice spacing: $|f(n+1) - f(n)| \sim |F'(x_n)| \, dx$

Primer on Lattice Techniques

"Good" lattice spacing: $|f(n+1) - f(n)| \sim |F'(x_n)|\, dx$

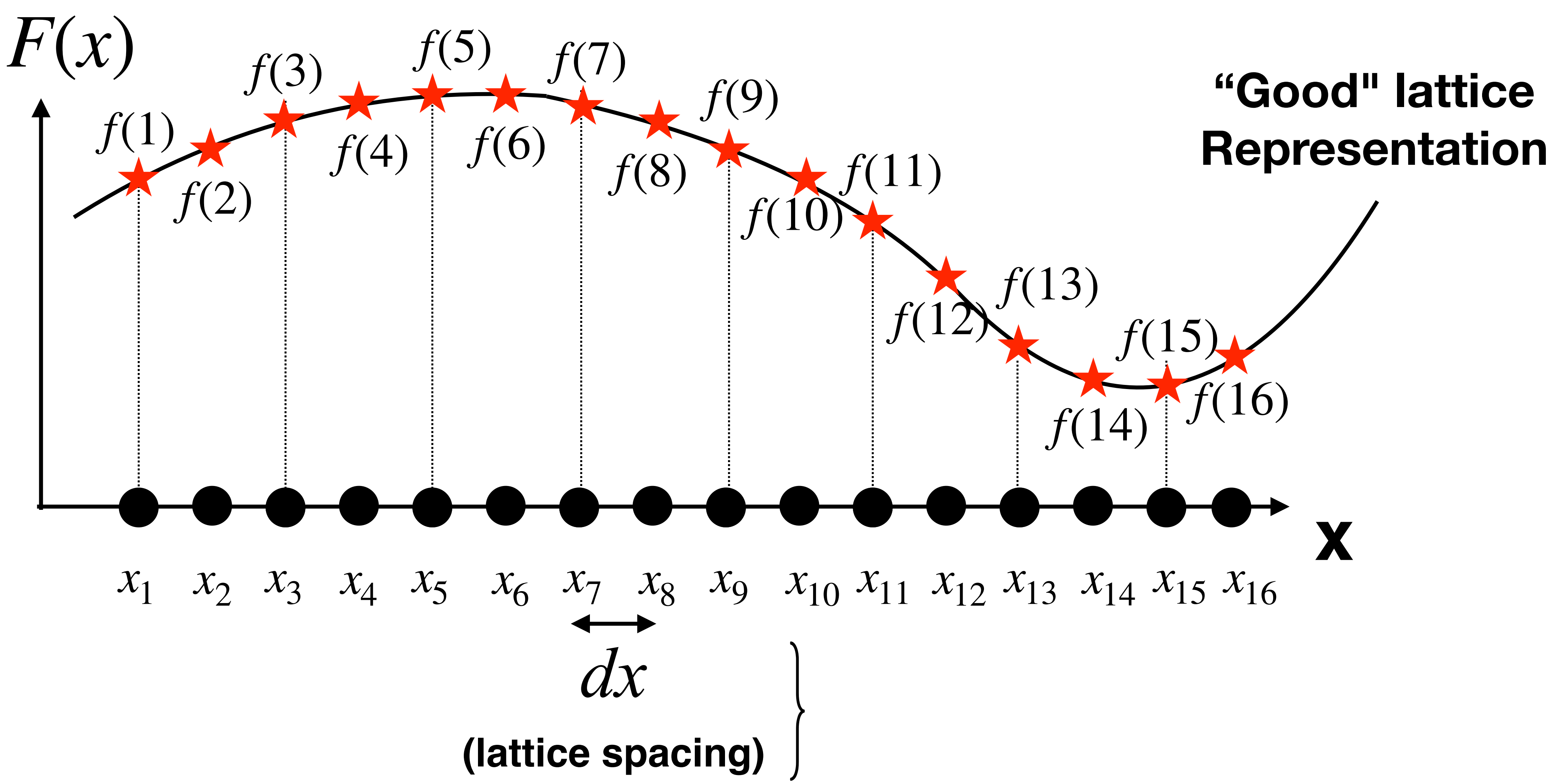
Primer on Lattice Techniques

"Good" lattice spacing: i.e. represent well your derivatives !

Primer on Lattice Techniques

"Good" lattice spacing: i.e. represent well your derivatives !

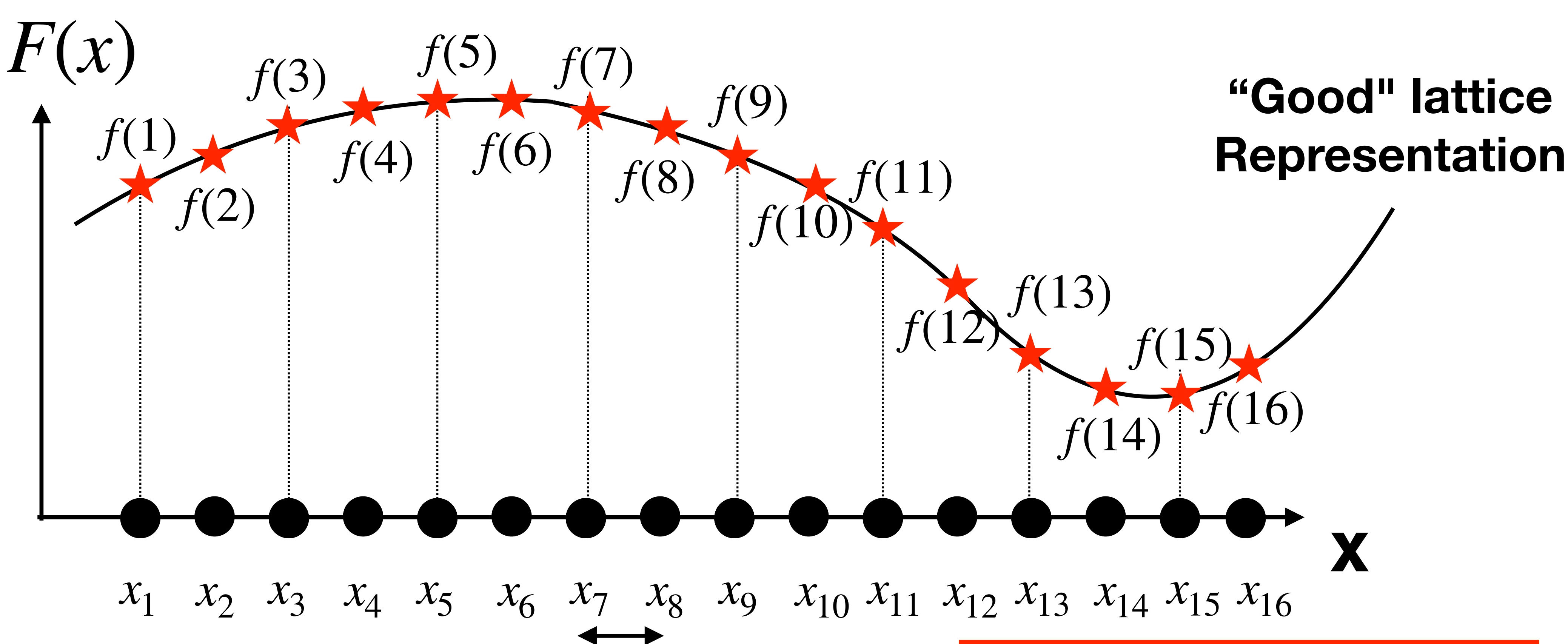
$$f(n) \equiv F(x_n \equiv x_* + n dx)$$



Primer on Lattice Techniques

"Good" lattice spacing: i.e. represent well your derivatives !

$$f(n) \equiv F(x_n \equiv x_* + n\,dx)$$



dx **(lattice spacing)** $\Longrightarrow$

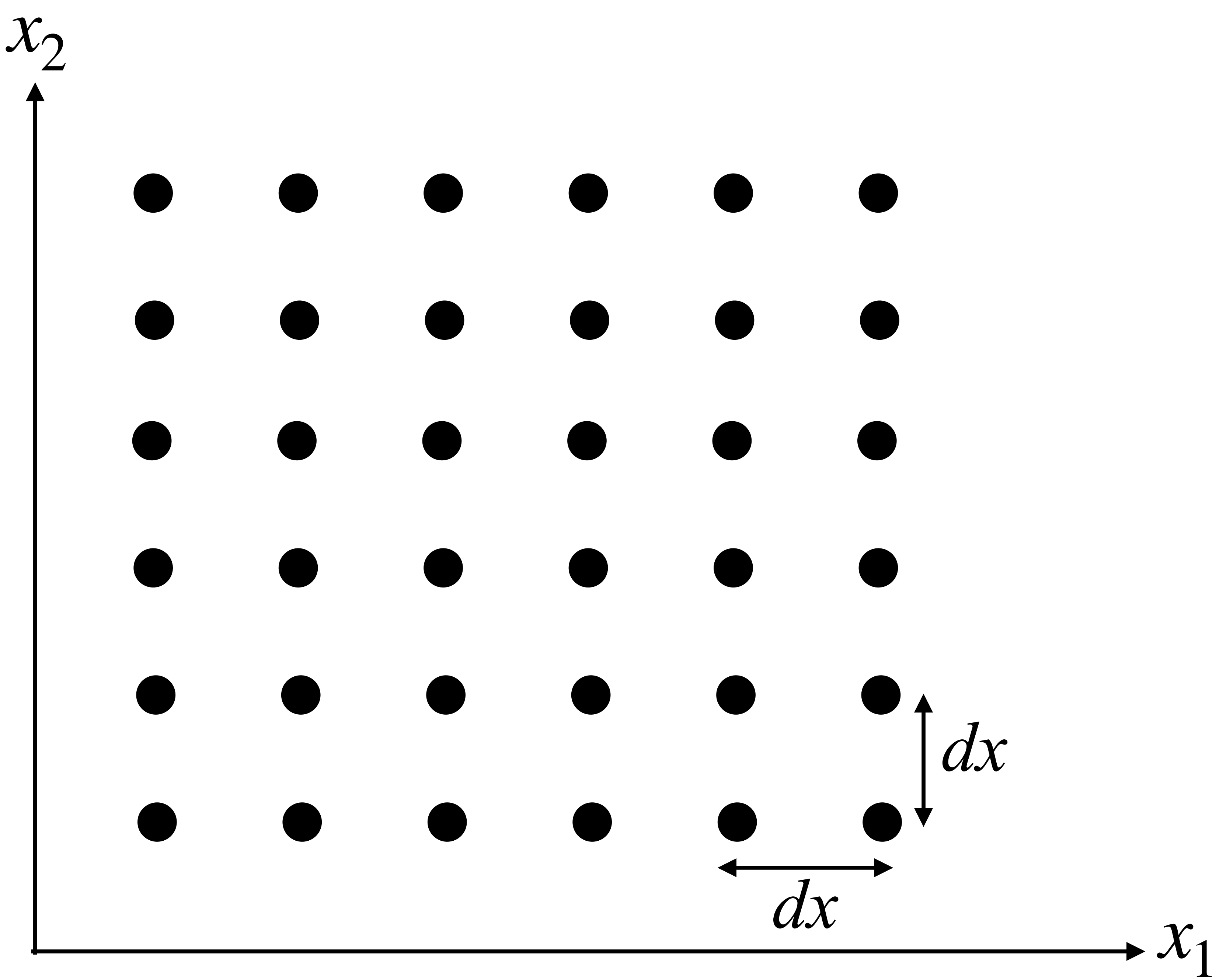
$$\frac{f(n+1) - f(n)}{dx} \approx F'(x_{n+1/2})$$

$$x_{n+1/2} \equiv \frac{x_n + x_{n+1}}{2}$$

more on this later !

Primer on Lattice Techniques

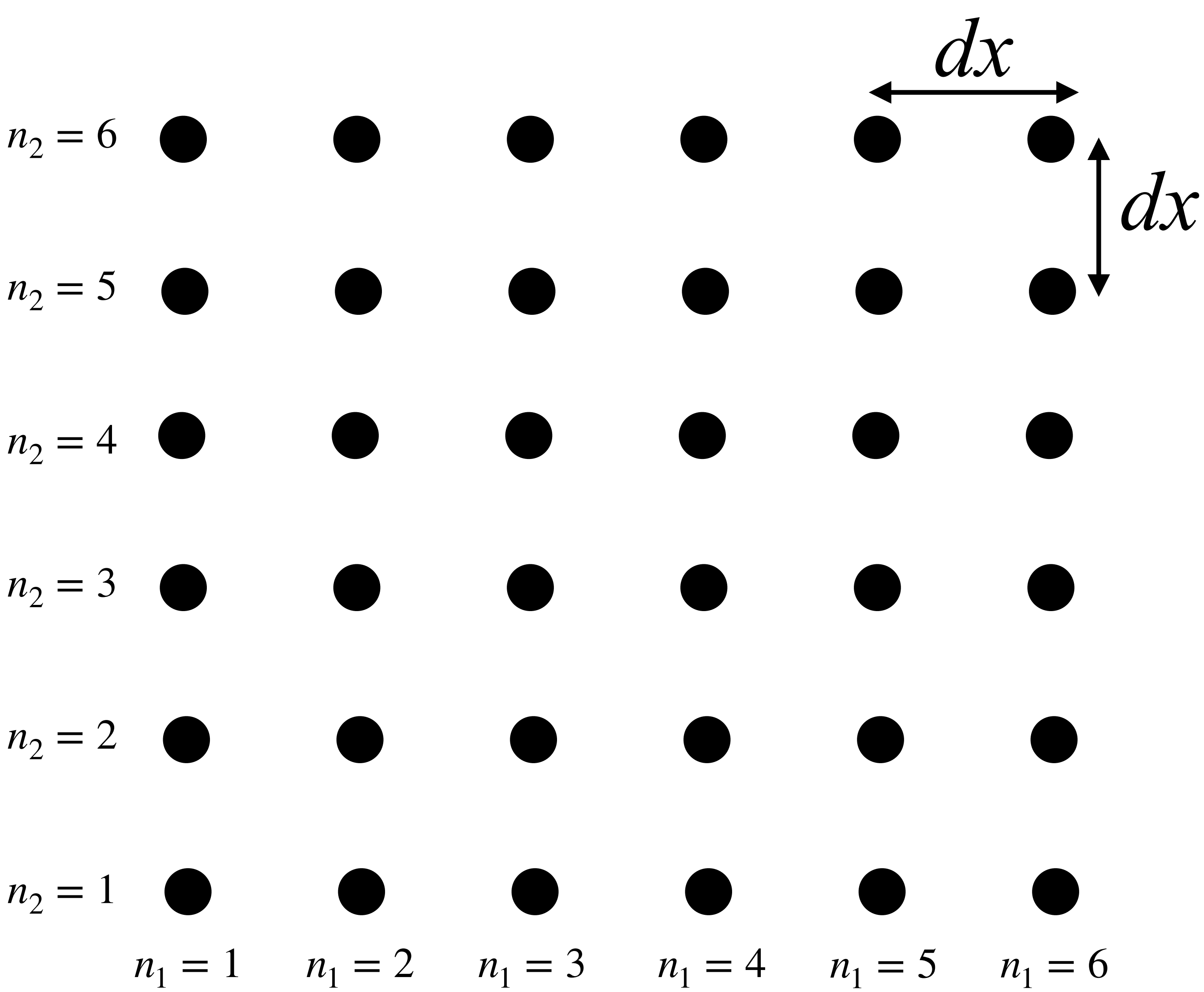
Generalization to 2 spatial dimensions (2D)



Primer on Lattice Techniques

Generalization to 2 spatial dimensions (2D)

$$\{\mathbf{x}_{n_1 n_2}\}\,,\; n_i = 1,\, 2,\, \ldots\ldots,\, N\,;\;\; i = 1{,}2 \qquad (N^2 \text{ entries})$$

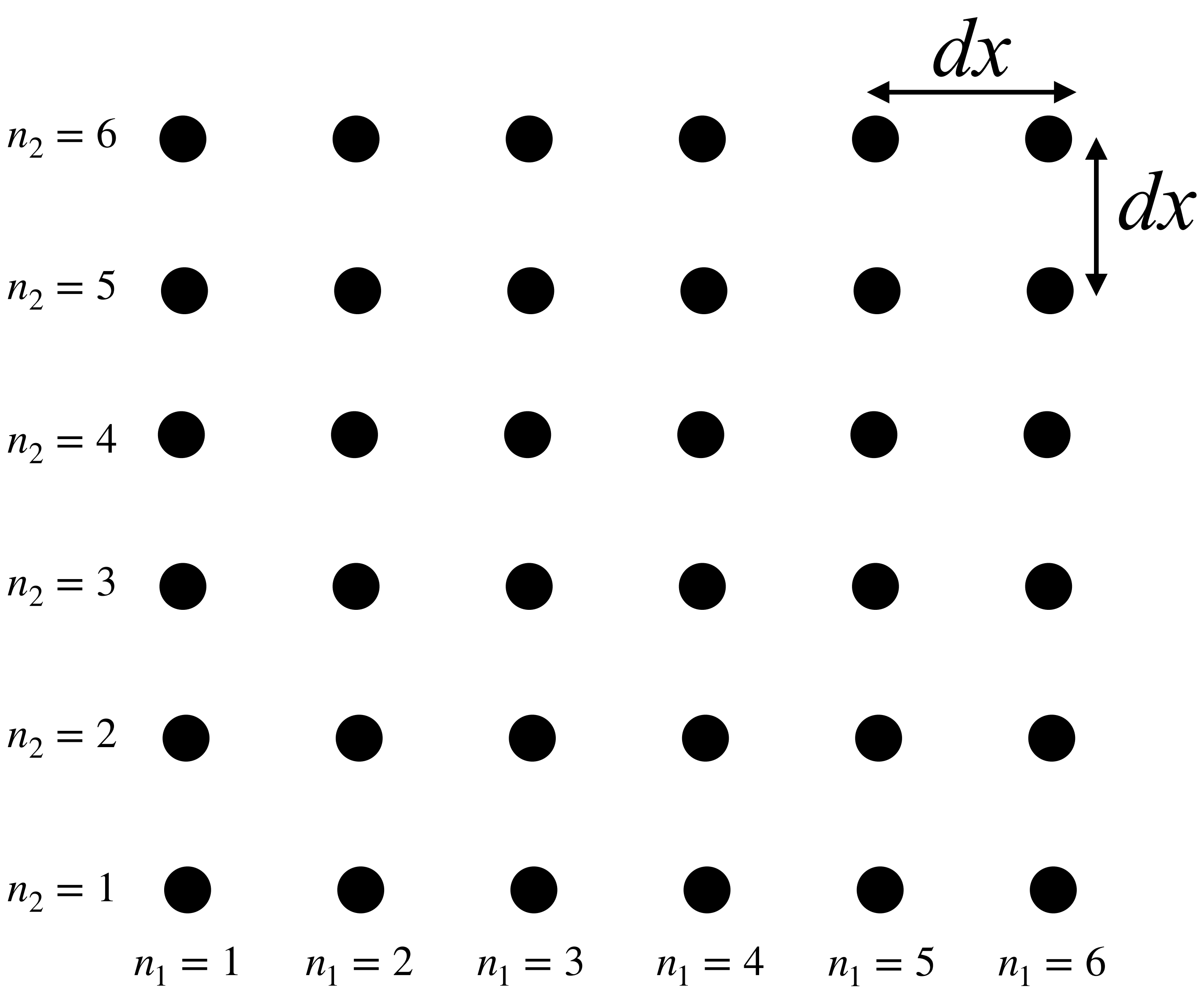


Primer on Lattice Techniques

Generalization to 2 spatial dimensions (2D)

$$\{\mathbf{x}_{n_1 n_2}\}\ ,\ n_i = 1,\ 2,\ \ldots\ldots,\ N;\ \ i = 1{,}2 \qquad (N^2 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2) \equiv F(\mathbf{x}_{n_1 n_2})$$

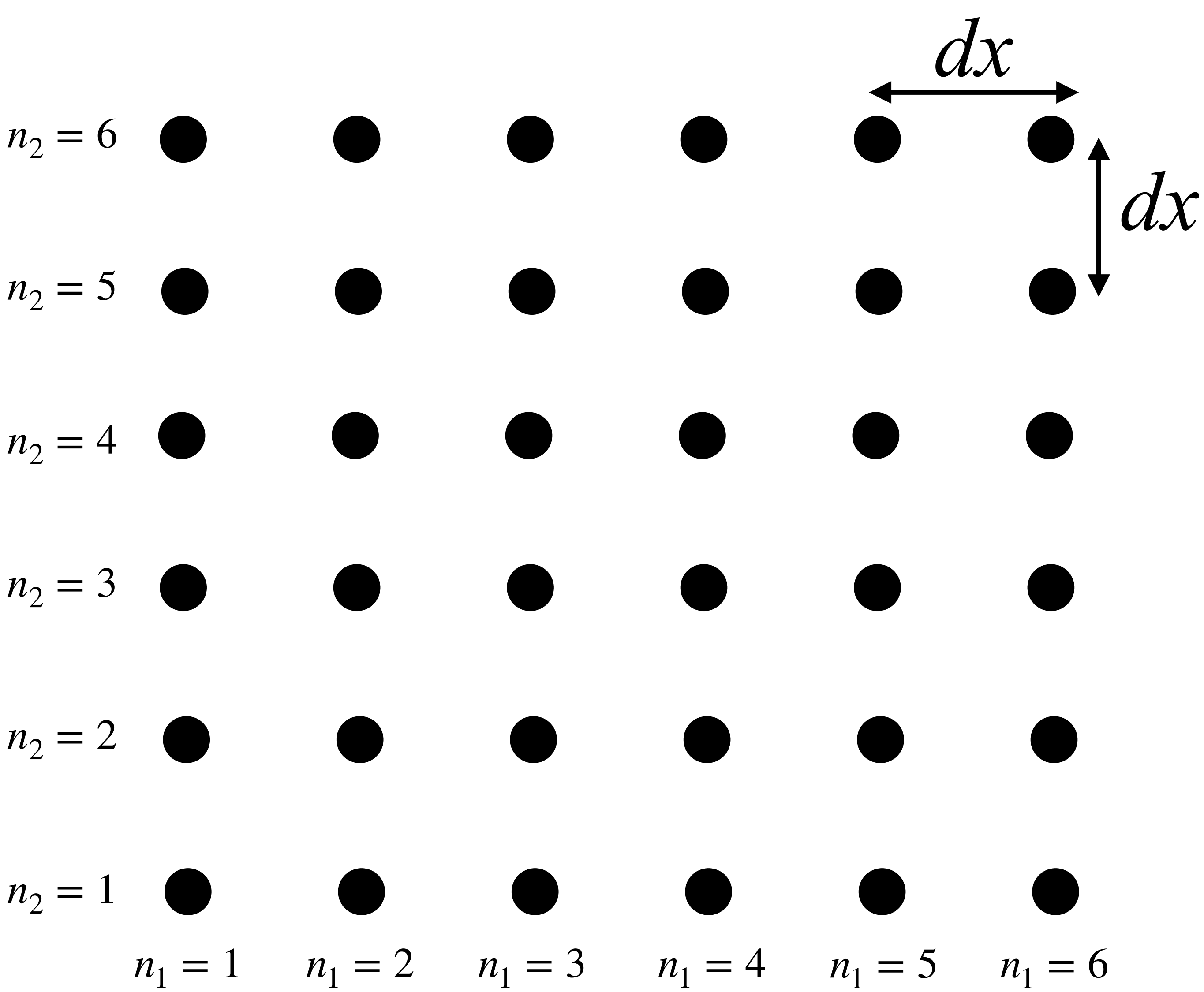


Primer on Lattice Techniques

Generalization to 2 spatial dimensions (2D)

$$\{\mathbf{x}_{n_1 n_2}\}\ ,\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2 \qquad (N^2 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2) \equiv F(\mathbf{x}_{n_1 n_2})$$



$$\{f(\mathbf{n})\}\ ,\quad N^2 \text{ entries}$$

$$\mathbf{n} \equiv (n_1, n_2)$$

$$(n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2)$$

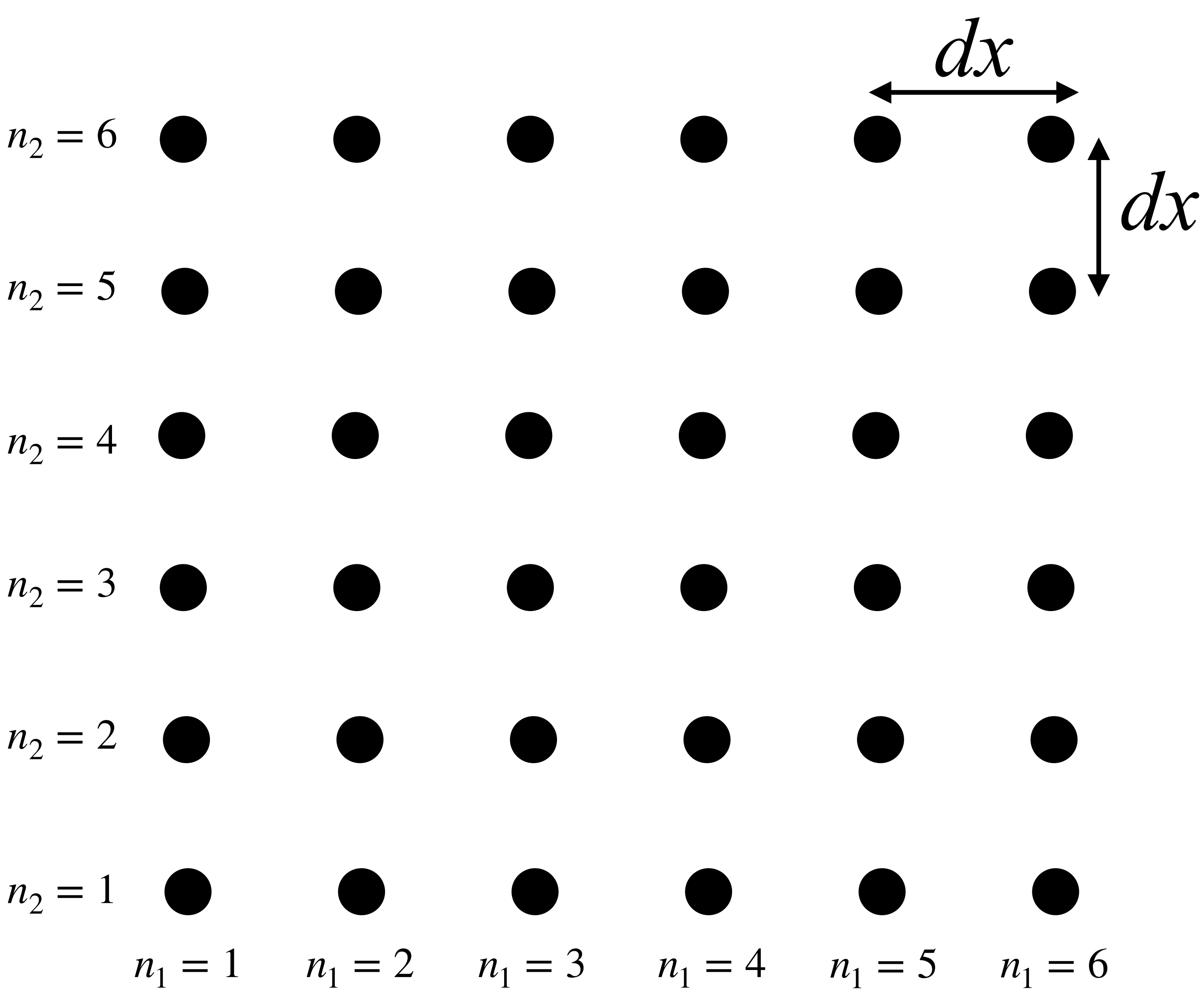
F(x) — Lattice representation

Primer on Lattice Techniques

Generalization to 2 spatial dimensions (2D)

$$\{\mathbf{x}_{n_1 n_2}\}\ ,\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1{,}2 \qquad (N^2 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2) \equiv F(\mathbf{x}_{n_1 n_2})$$



$$\{f(\mathbf{n})\}\ ,\quad N^2 \text{ entries}$$

$$\mathbf{n} \equiv (n_1, n_2)$$

$$(n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1{,}2)$$

"Good" lattice spacing IF:

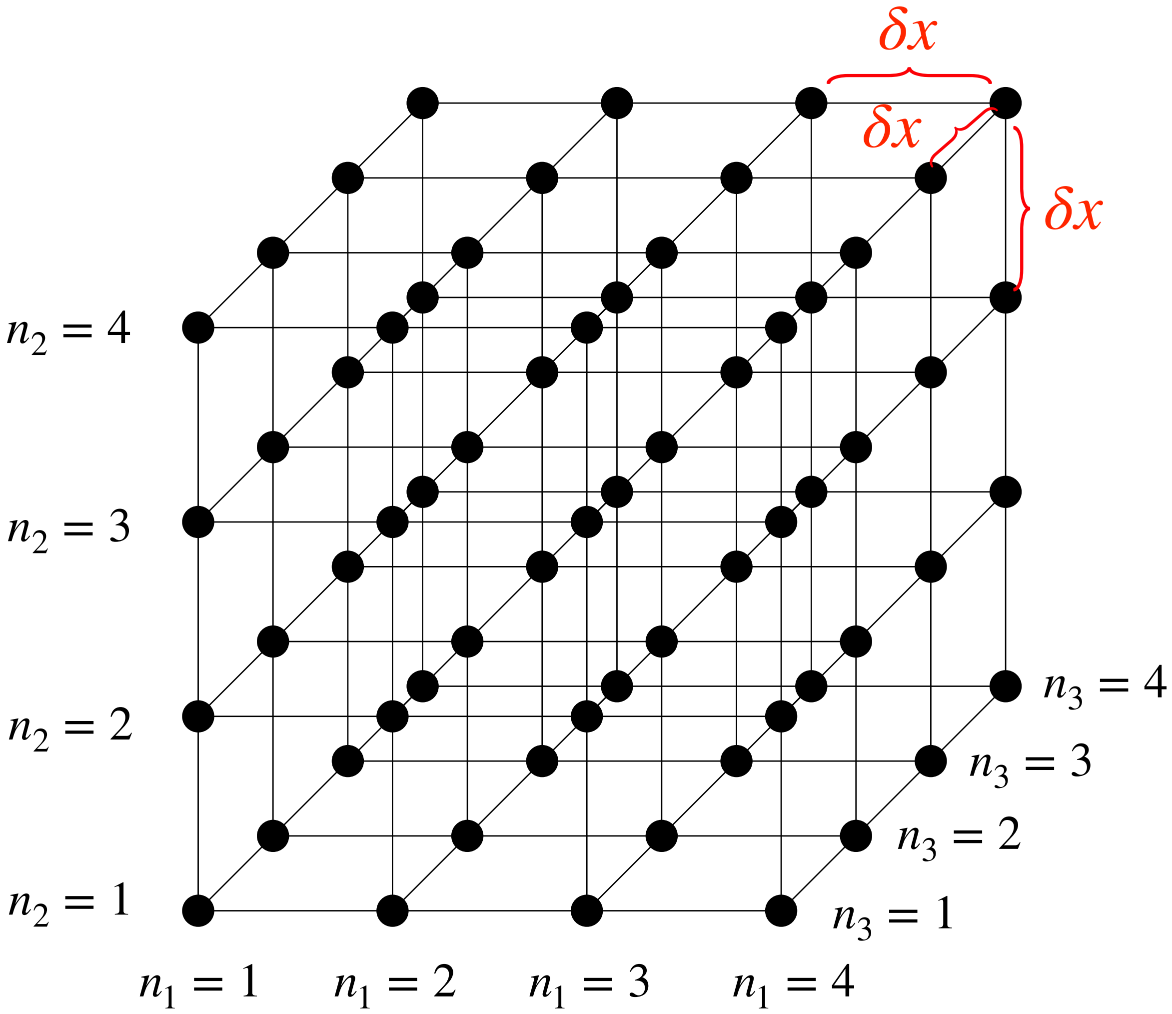
$$\frac{f(\mathbf{n}+\hat{\imath}) - f(\mathbf{n})}{dx} \approx \left.\frac{\partial F}{\partial x_i}\right|_{(\mathbf{n}+\hat{\imath}/2)dx}$$

$\hat{\imath} \equiv$ **unit vector in *i*-direction**

(i = 1,2,3)

Primer on Lattice Techniques

Generalization to 3 spatial dimensions (3D)

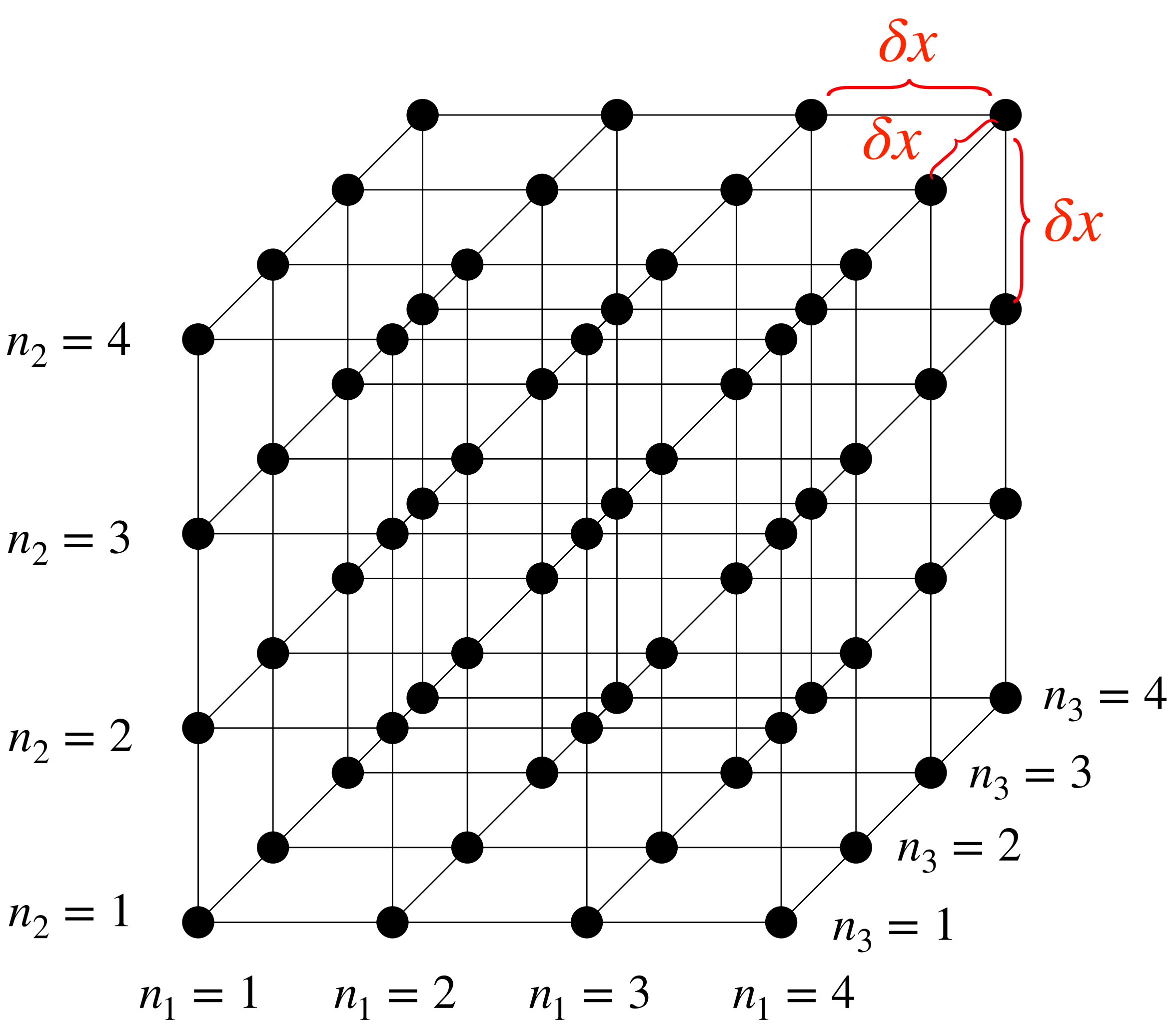


Primer on Lattice Techniques

Generalization to 3 spatial dimensions (3D)

$$\{\mathbf{x}_{n_1 n_2 n_3}\}\,,\; n_i = 1,\, 2,\, \ldots.,\, N\,;\;\; i = 1,2,3 \quad (N^3 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2, n_3) \equiv F(\mathbf{x}_{n_1 n_2 n_3})$$

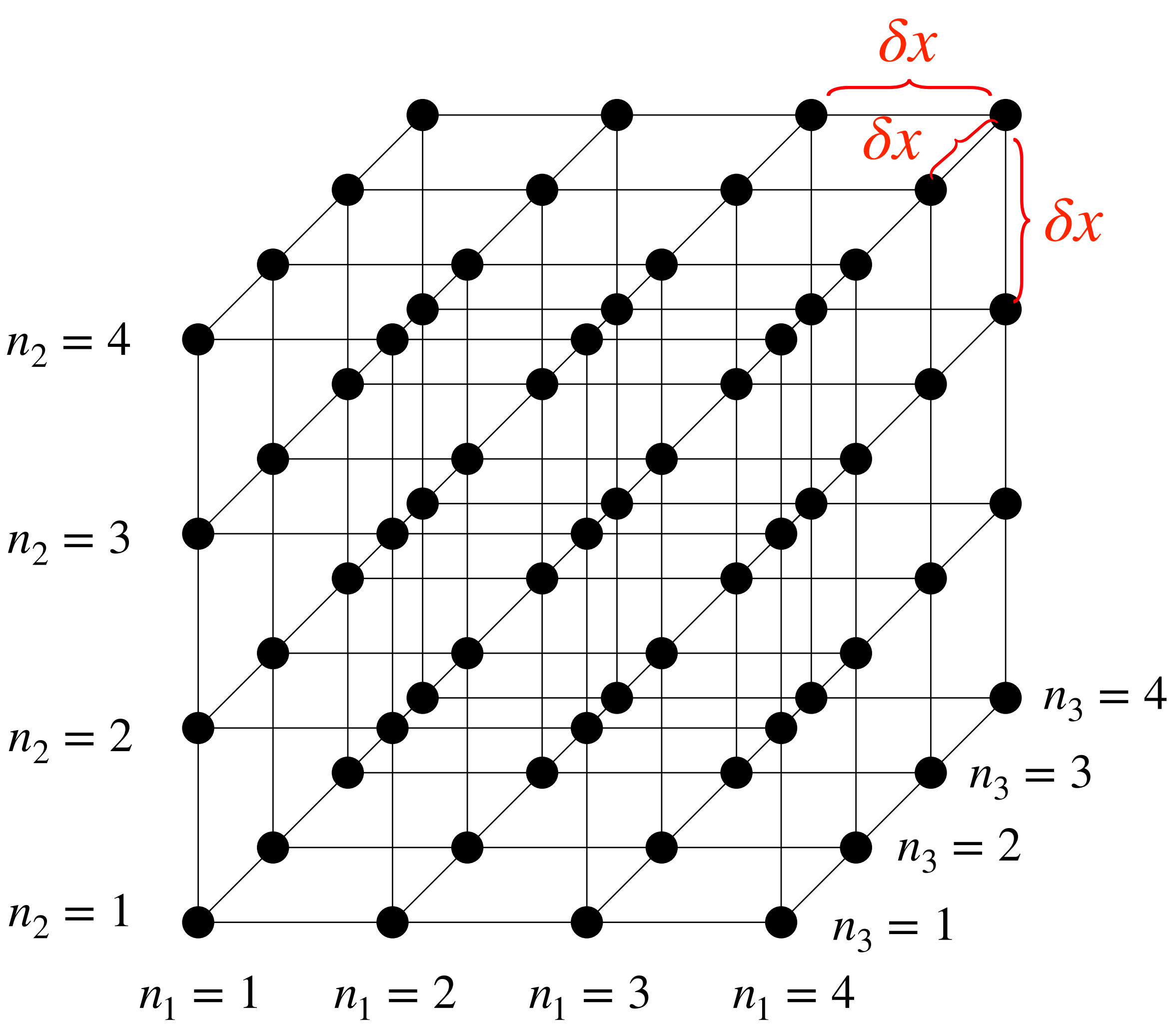


Primer on Lattice Techniques

Generalization to 3 spatial dimensions (3D)

$$\{\mathbf{x}_{n_1 n_2 n_3}\}\ ,\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2,3 \quad (N^3 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2, n_3) \equiv F(\mathbf{x}_{n_1 n_2 n_3})$$



$$\{f(\mathbf{n})\}\ ,\quad N^3 \text{ entries}$$

$$\mathbf{n} \equiv (n_1, n_2, n_3)$$

$$(n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2,3)$$

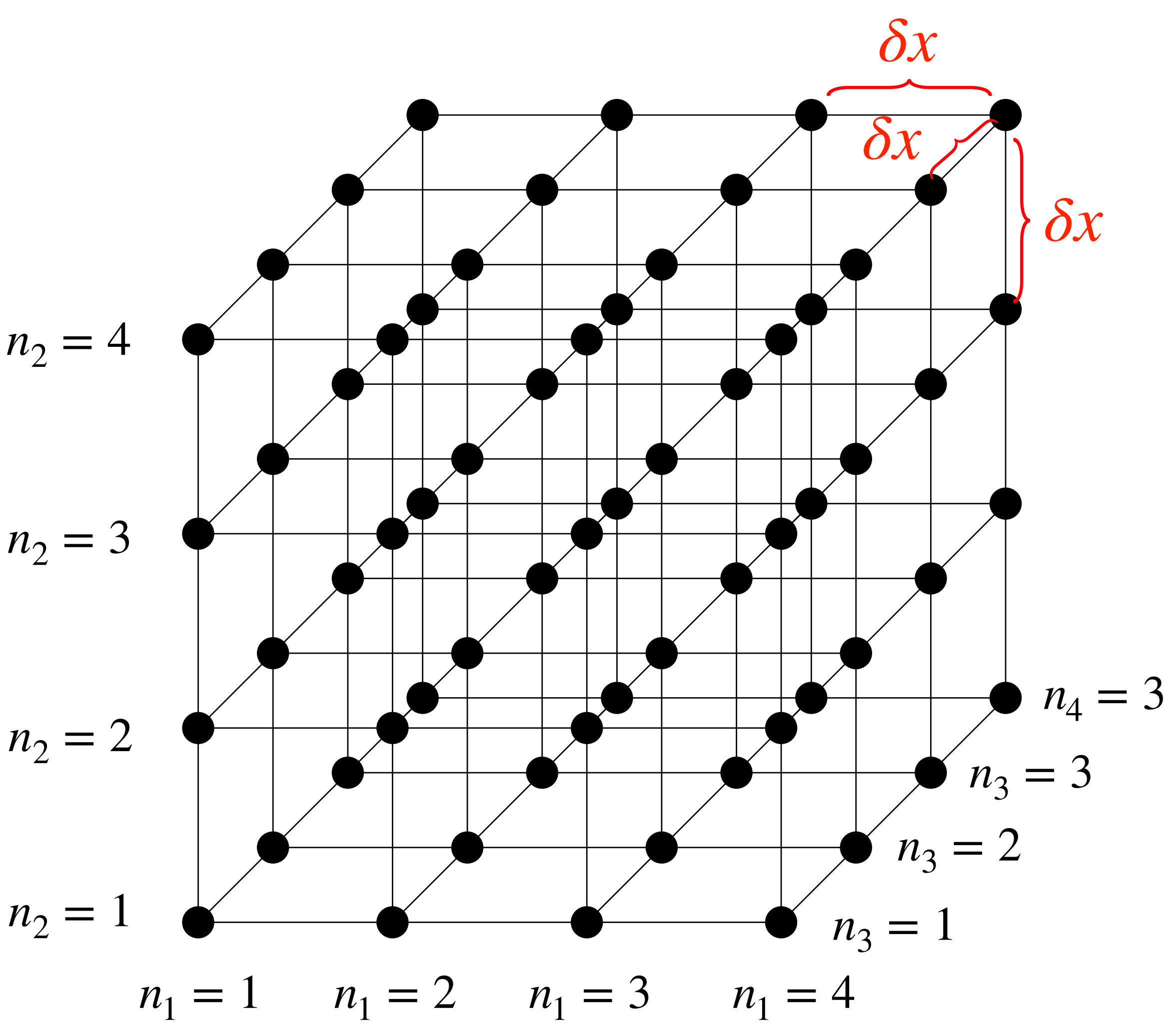
F(x) — Lattice representation

Primer on Lattice Techniques

Generalization to 3 spatial dimensions (3D)

$$\{\mathbf{x}_{n_1 n_2 n_3}\}\,,\; n_i = 1,\,2,\,\ldots\ldots,\,N\,;\;\; i = 1,2,3 \quad (N^3 \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2, n_3) \equiv F(\mathbf{x}_{n_1 n_2 n_3})$$



$$\{f(\mathbf{n})\}\,,\quad N^3 \text{ entries}$$

$$\mathbf{n} \equiv (n_1, n_2, n_3)$$

$$(n_i = 1,\,2,\,\ldots\ldots,\,N\,;\;\; i = 1,2,3)$$

"Good" lattice spacing IF:

$$\frac{f(\mathbf{n}+\hat{i}) - f(\mathbf{n})}{dx} \approx \left.\frac{\partial F}{\partial x_i}\right|_{(\mathbf{n}+\hat{i}/2)dx}$$

$\hat{i}$ ≡ **unit vector in *i*-direction**

(i = 1,2,3)

Primer on Lattice Techniques

Generalization to d-spatial dimensions (d-D)

$$\{\mathbf{x}_{n_1 n_2 \ldots n_d}\}\,,\; n_i = 1,\,2,\,\ldots\ldots,\,N\,;\;\; i = 1,2,\ldots,d \qquad (N^d \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2, \ldots, n_d) \equiv F(\mathbf{x}_{n_1 n_2 \ldots n_d})$$

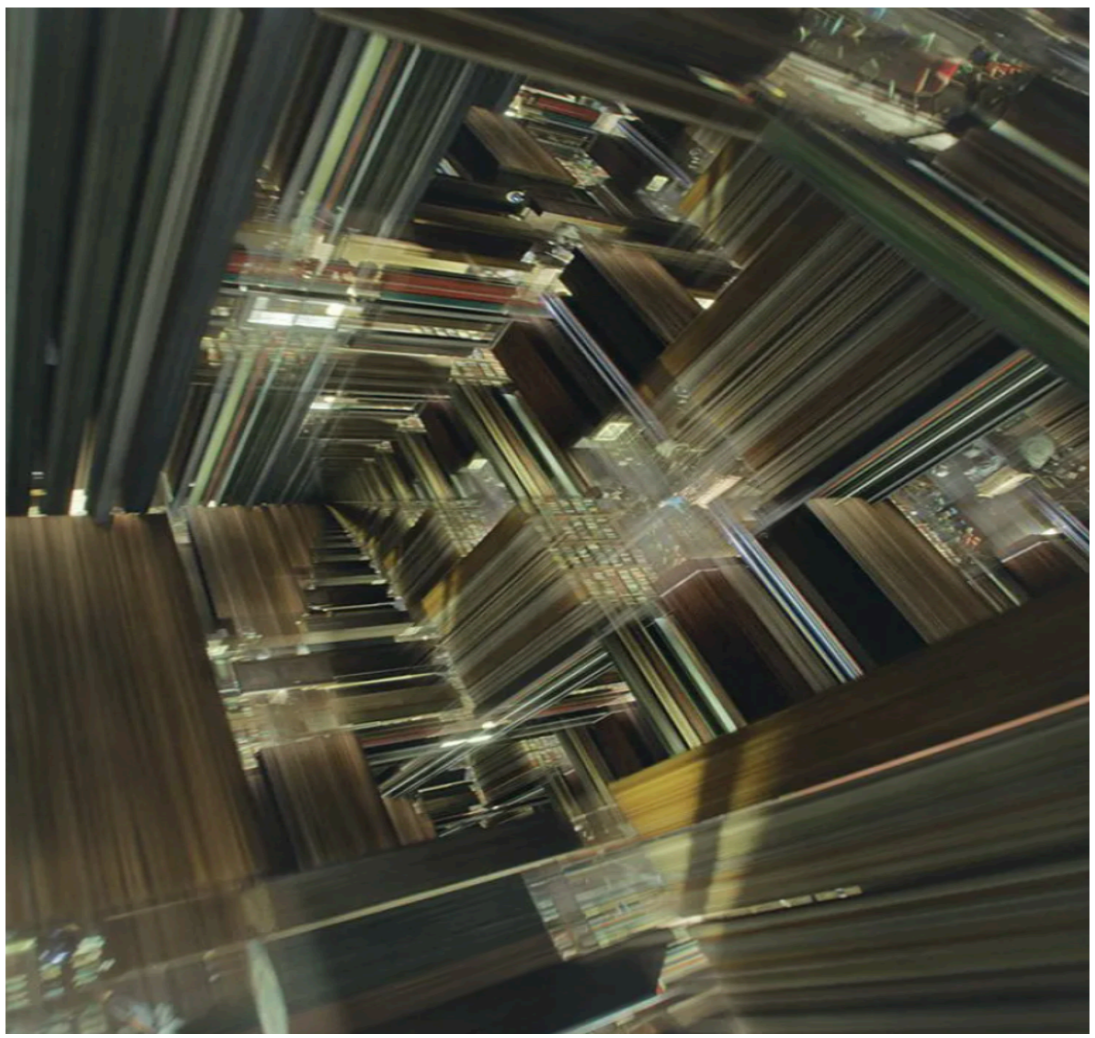
(4D lattice according to Christopher Nolan)

Primer on Lattice Techniques

Generalization to d-spatial dimensions (d-D)

$$\{\mathbf{x}_{n_1 n_2 \ldots n_d}\}\ ,\ n_i = 1,\ 2,\ \ldots.,\ N\ ;\ \ i = 1,2,\ldots,d \qquad (N^d \text{ entries})$$

$$F(\mathbf{x}) \quad \longrightarrow \quad f(n_1, n_2, \ldots, n_d) \equiv F(\mathbf{x}_{n_1 n_2 \ldots n_d})$$



(4D lattice according to Christopher Nolan)

F(x) — Lattice representation

$$\{f(\mathbf{n})\}\ ,\quad N^d \text{ entries}$$

$$\mathbf{n} \equiv (n_1, n_2, \ldots, n_d)$$

$$(n_i = 1,\ 2,\ \ldots.,\ N\ ;\ \ i = 1,2,\ldots,d)$$

"Good" lattice spacing IF:

$$\frac{f(\mathbf{n} + \hat{i}) - f(\mathbf{n})}{dx} \approx \left.\frac{\partial F}{\partial x_i}\right|_{(\mathbf{n}+\hat{i}/2)dx}$$

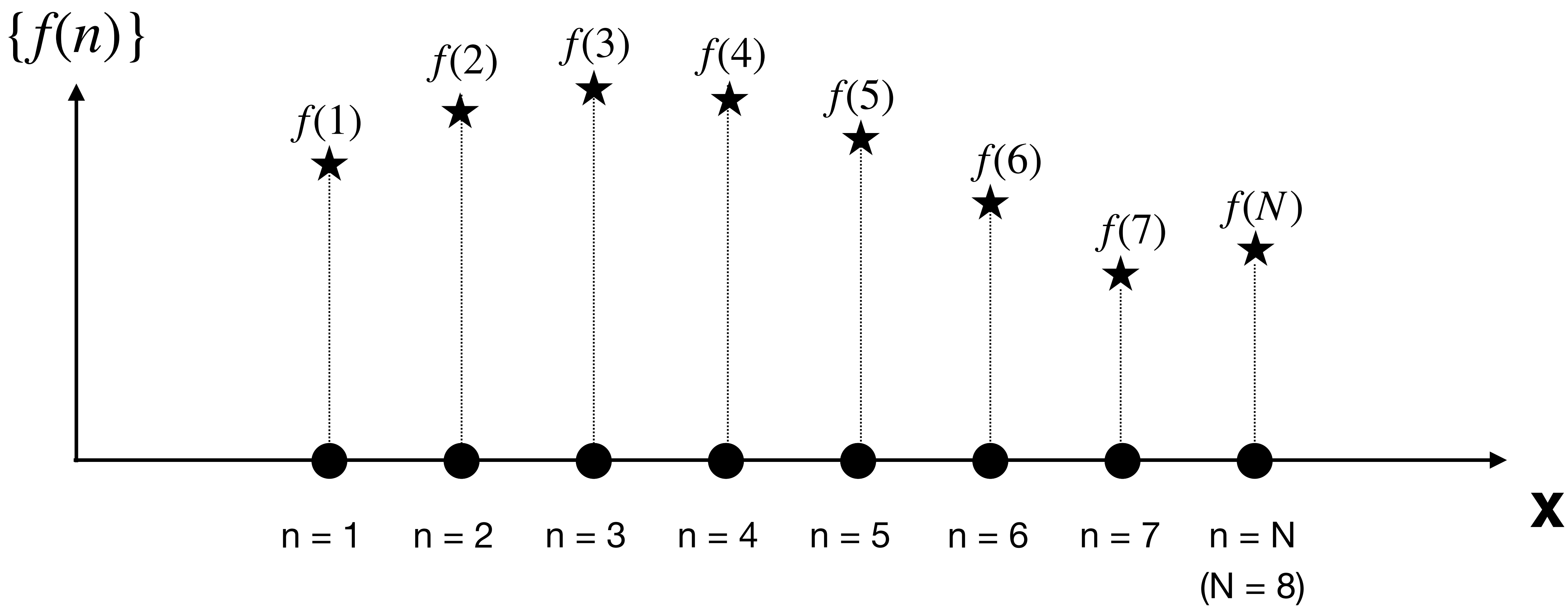
$\hat{i} \equiv$ **unit vector in *i*-direction**

Primer on Lattice Techniques

What about the boundaries ?

Primer on Lattice Techniques

What about the boundaries ?

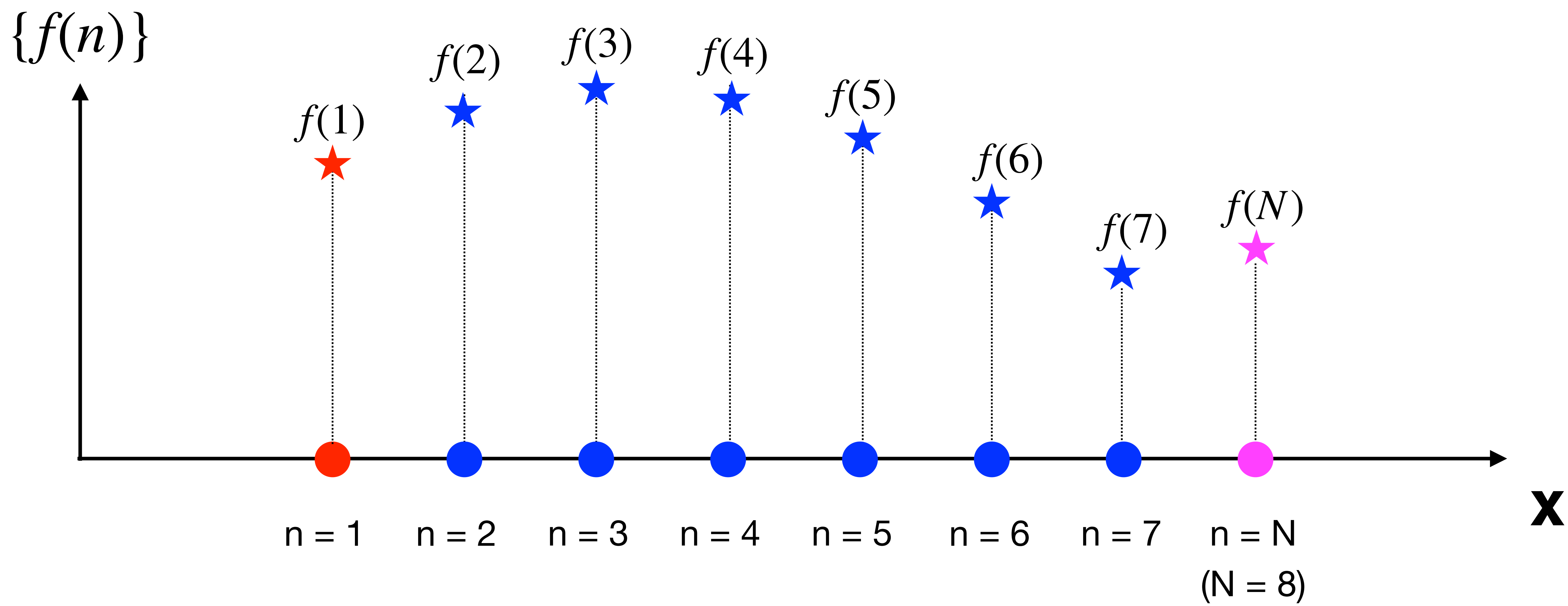


$$\{f(n) \equiv F(x_n)\}\ ,\ n = 1,\ 2,\ \ldots\ ,\ N$$

(example, N = 8)

Primer on Lattice Techniques

What about the boundaries ?



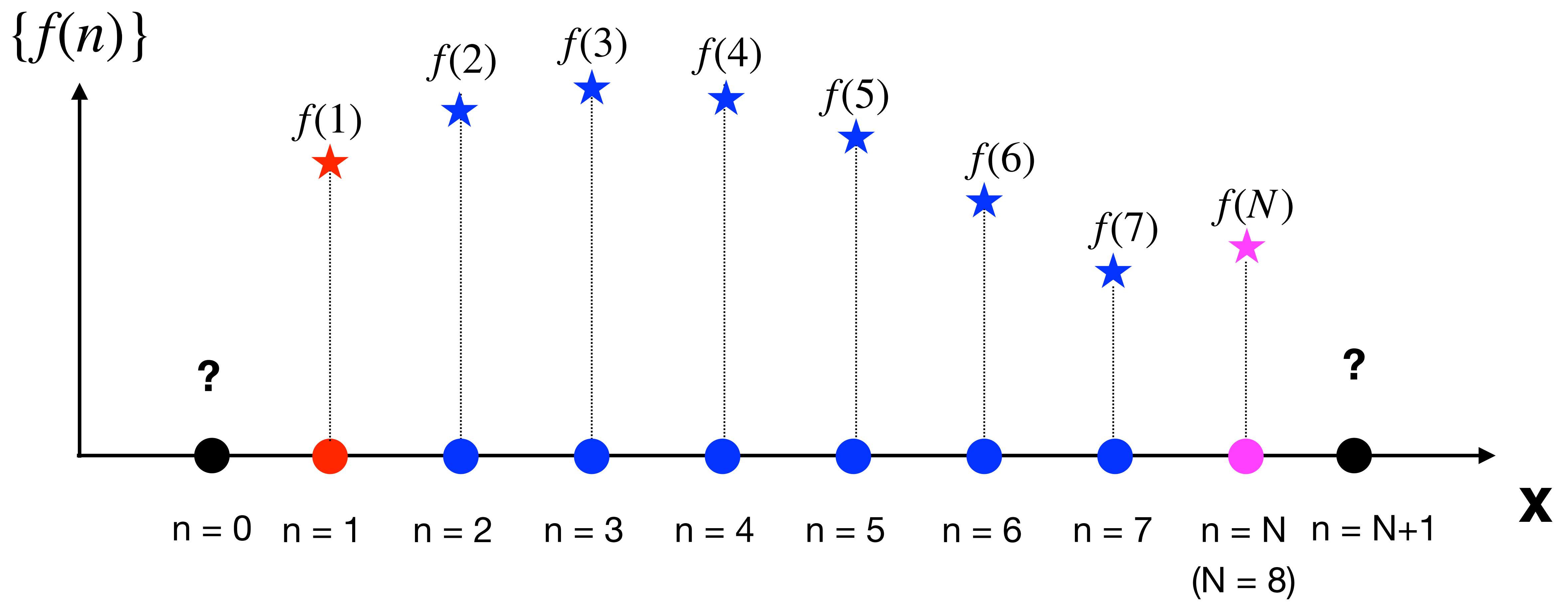
$$f(1) \equiv F(x_1)\ ;\ f(N) \equiv F(x_N)$$

$$f(n,t) \equiv F(x_n,t)\ ,n = 2,3,..,N-1$$

Fixed Boundary Conditions

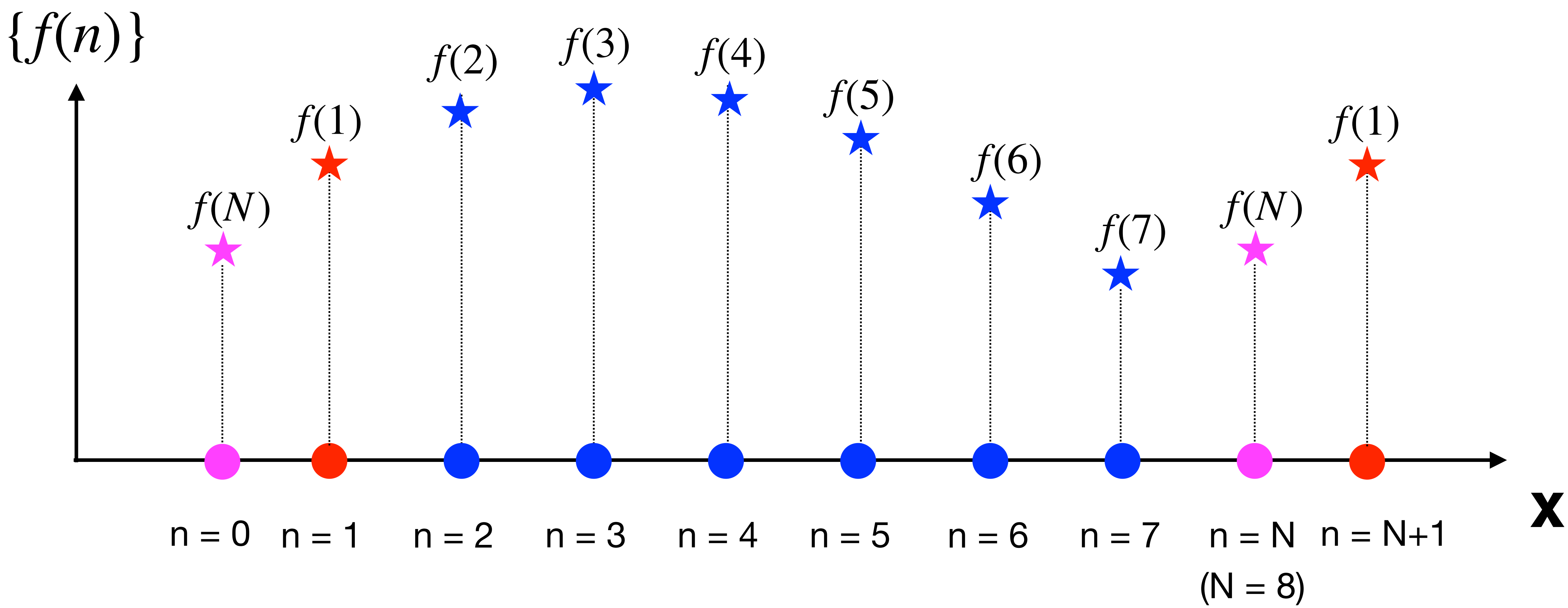
Primer on Lattice Techniques

What about the boundaries ?



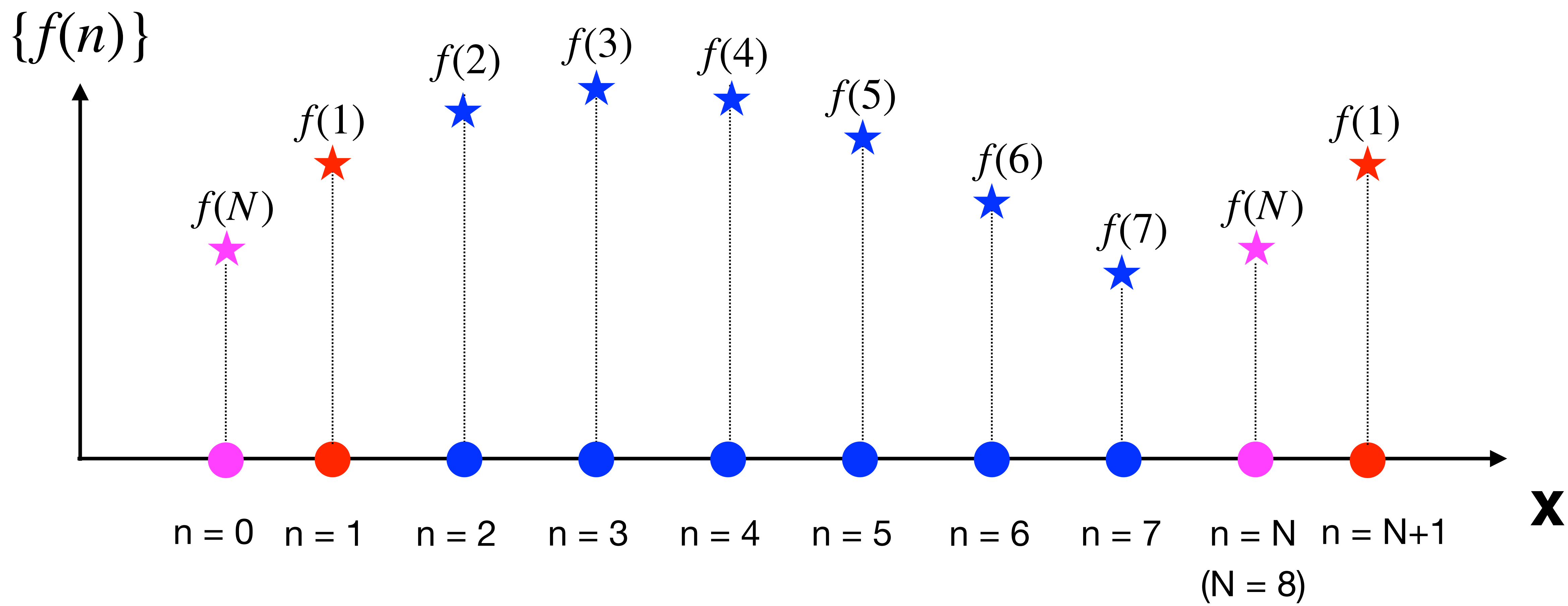
Primer on Lattice Techniques

What about the boundaries ?



Primer on Lattice Techniques

What about the boundaries ?



$$f(N+1) \equiv f(1) \;;\; f(0) \equiv f(N)$$

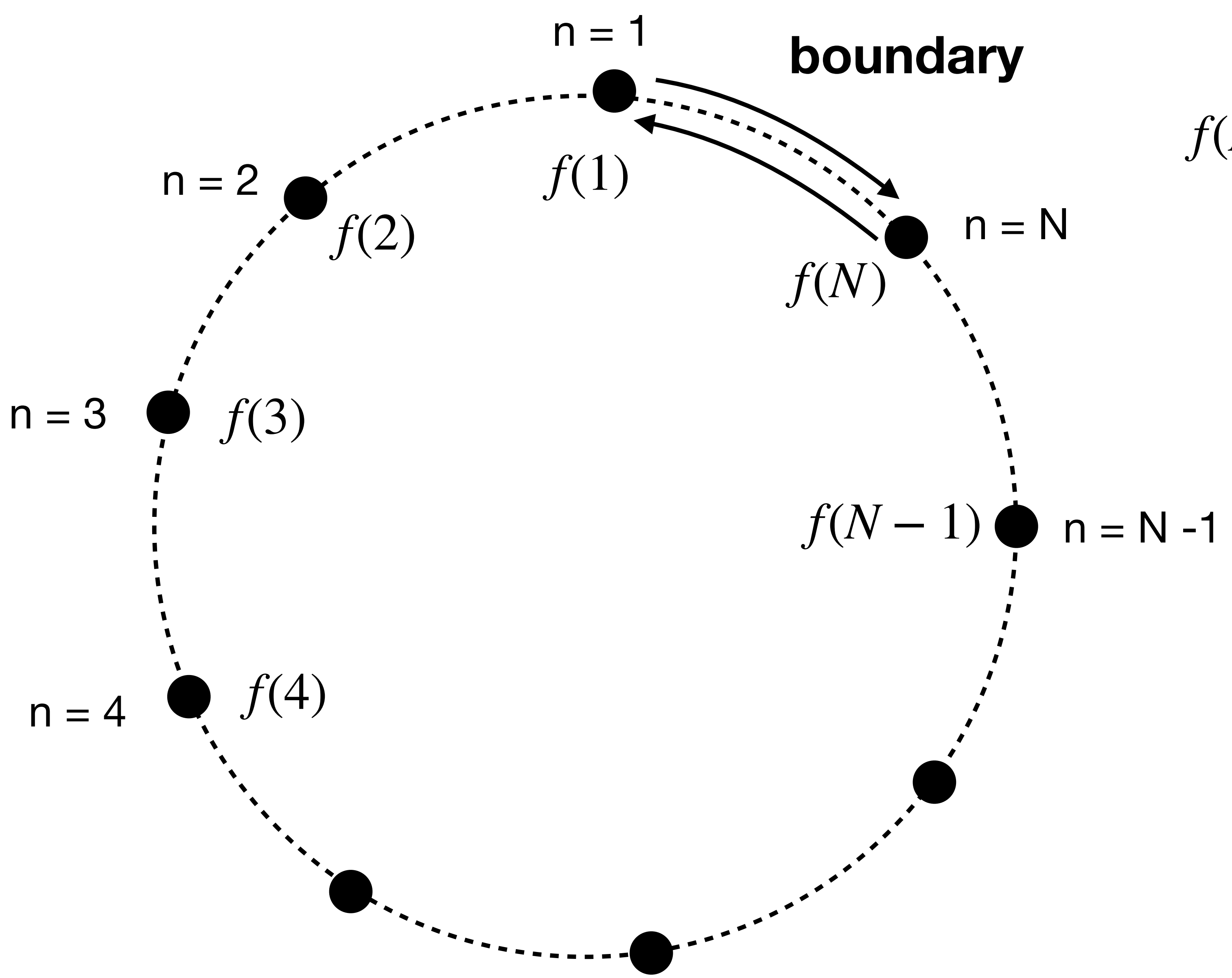
$$f(n,t) \equiv F(x_n,t) \;\; , n = 1,2,3,..,N$$

Periodic Boundary Conditions

Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 1D

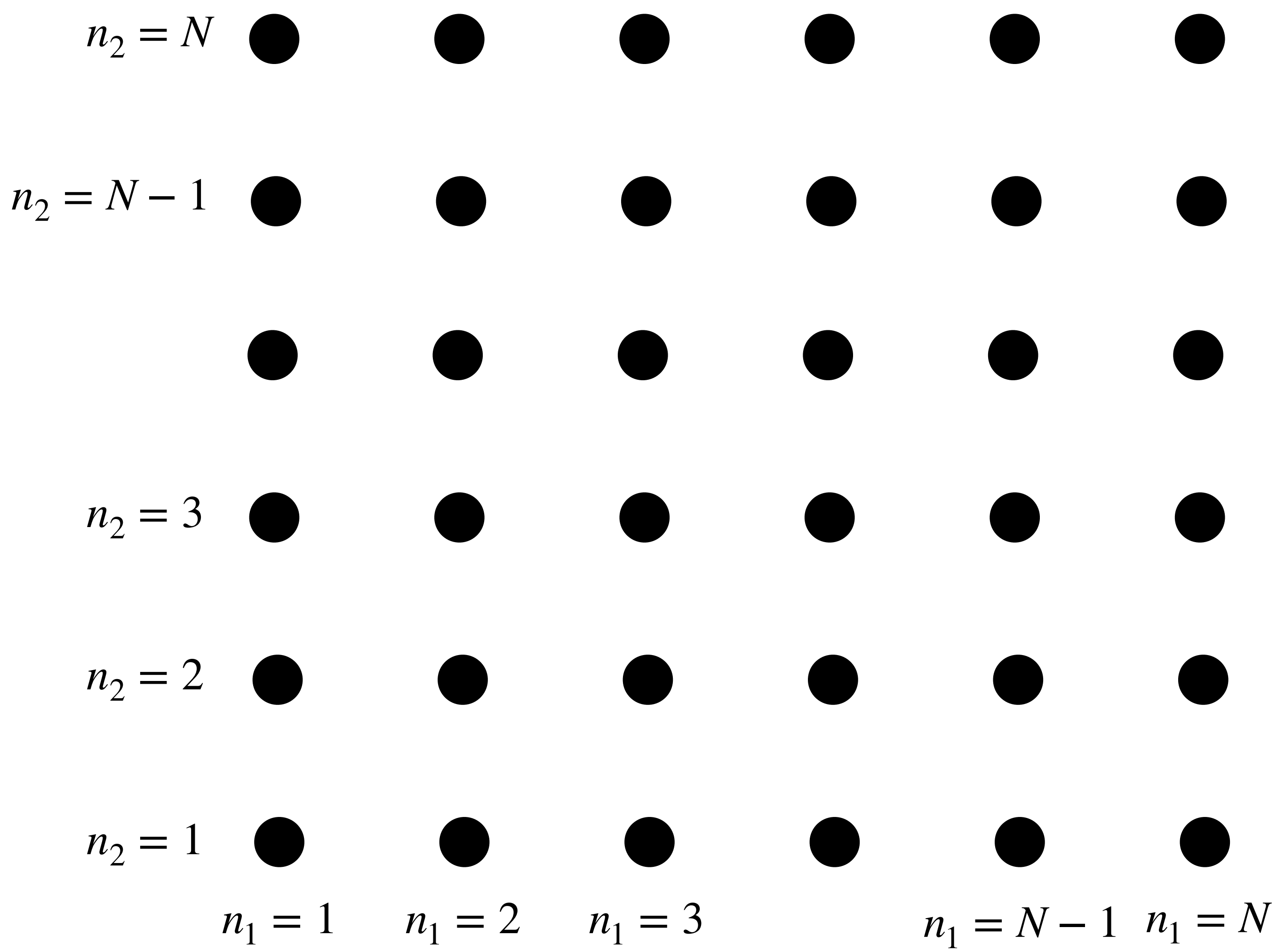


$$\left.\begin{aligned} f(N+1) &\equiv f(1) \\ f(0) &\equiv f(N) \end{aligned}\right\}$$

Primer on Lattice Techniques

What about the boundaries ?

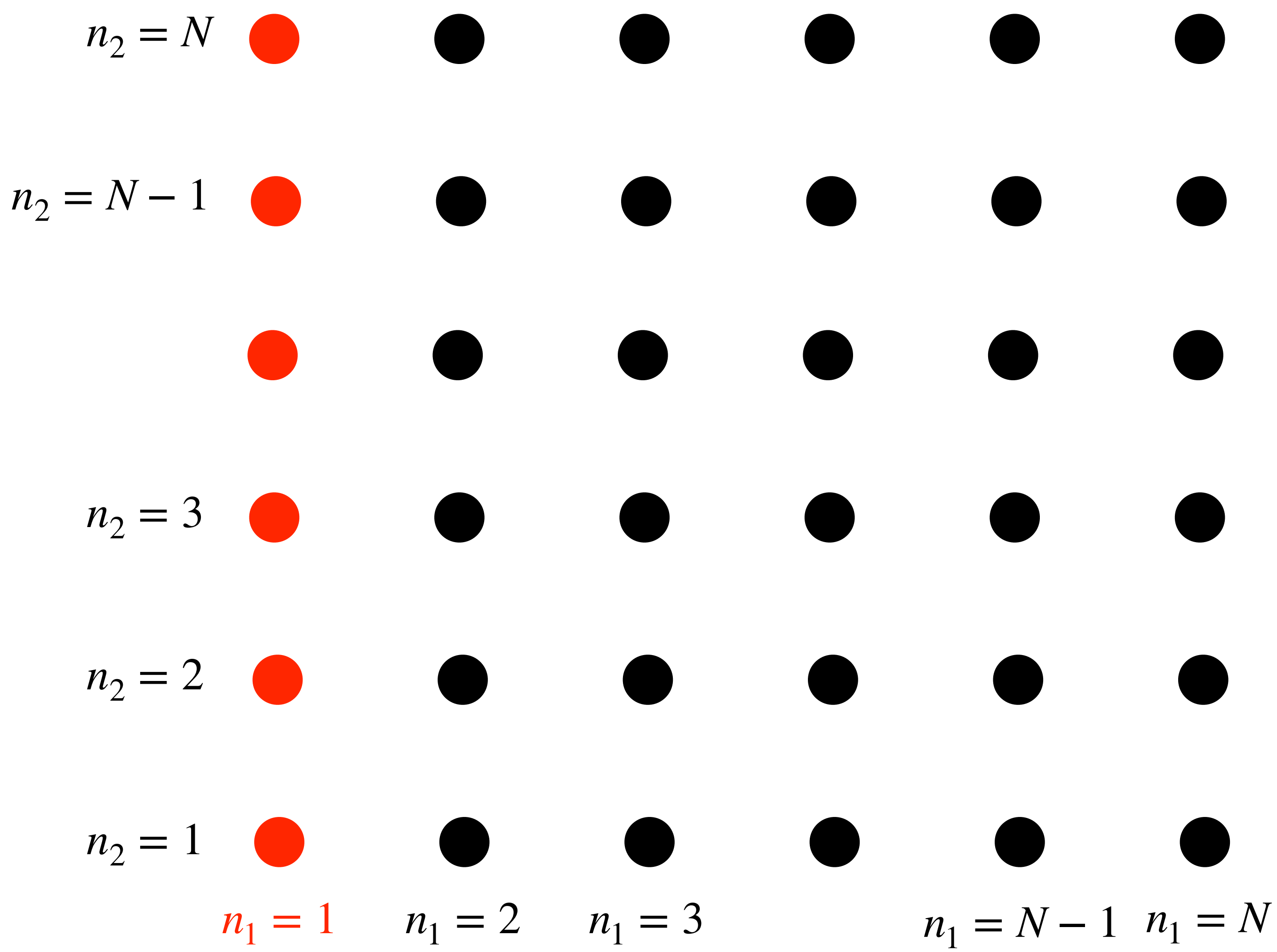
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

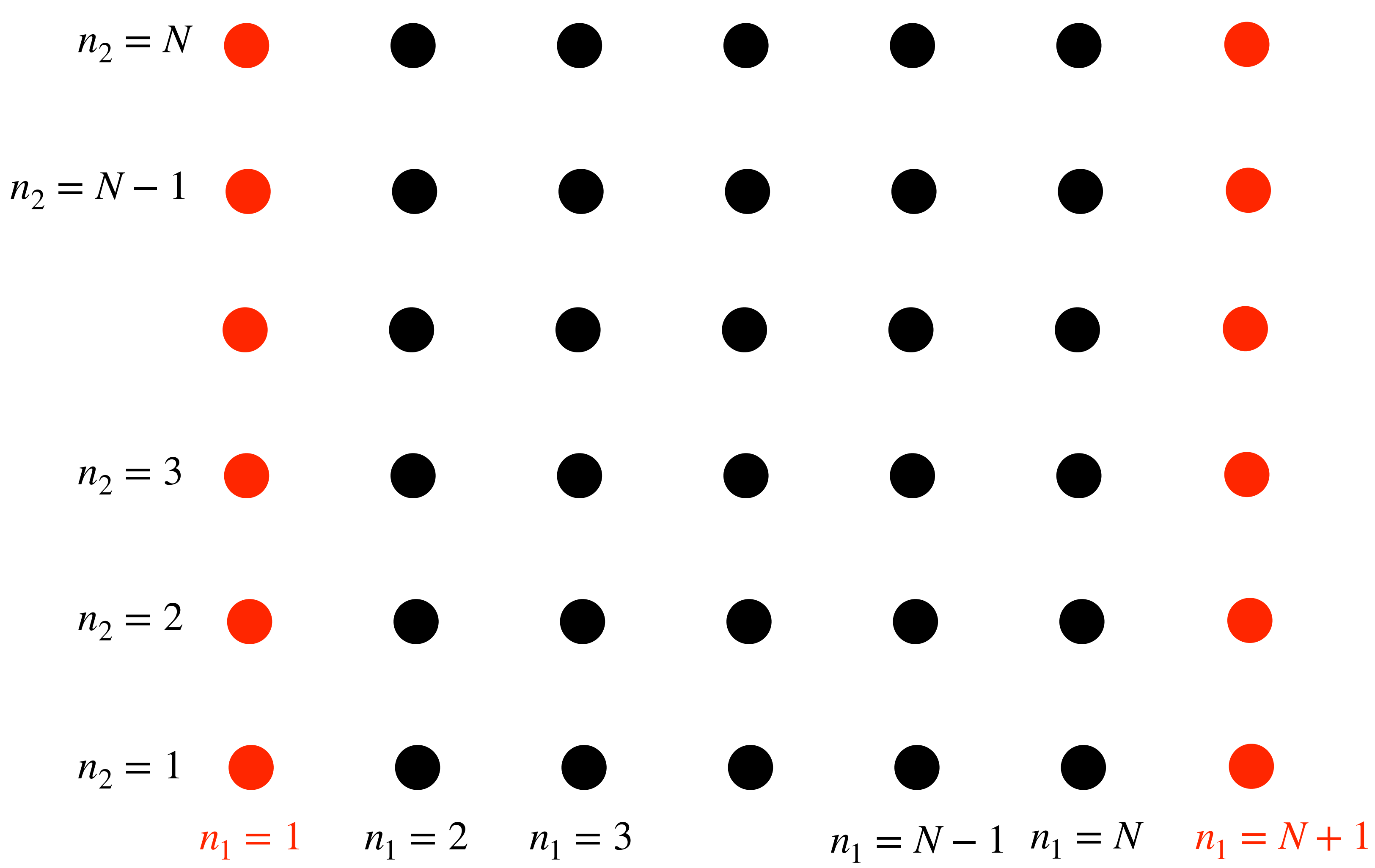
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

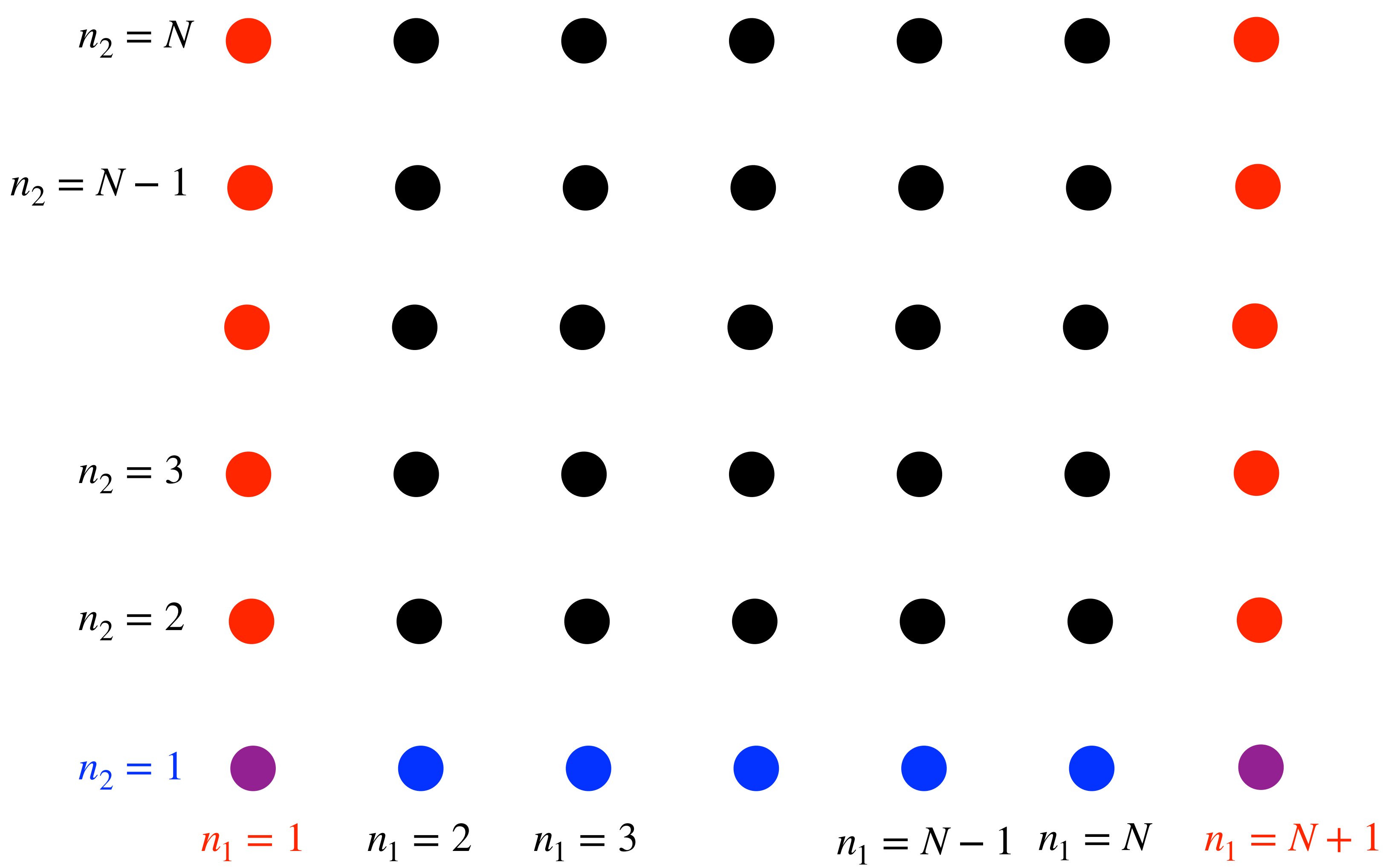
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

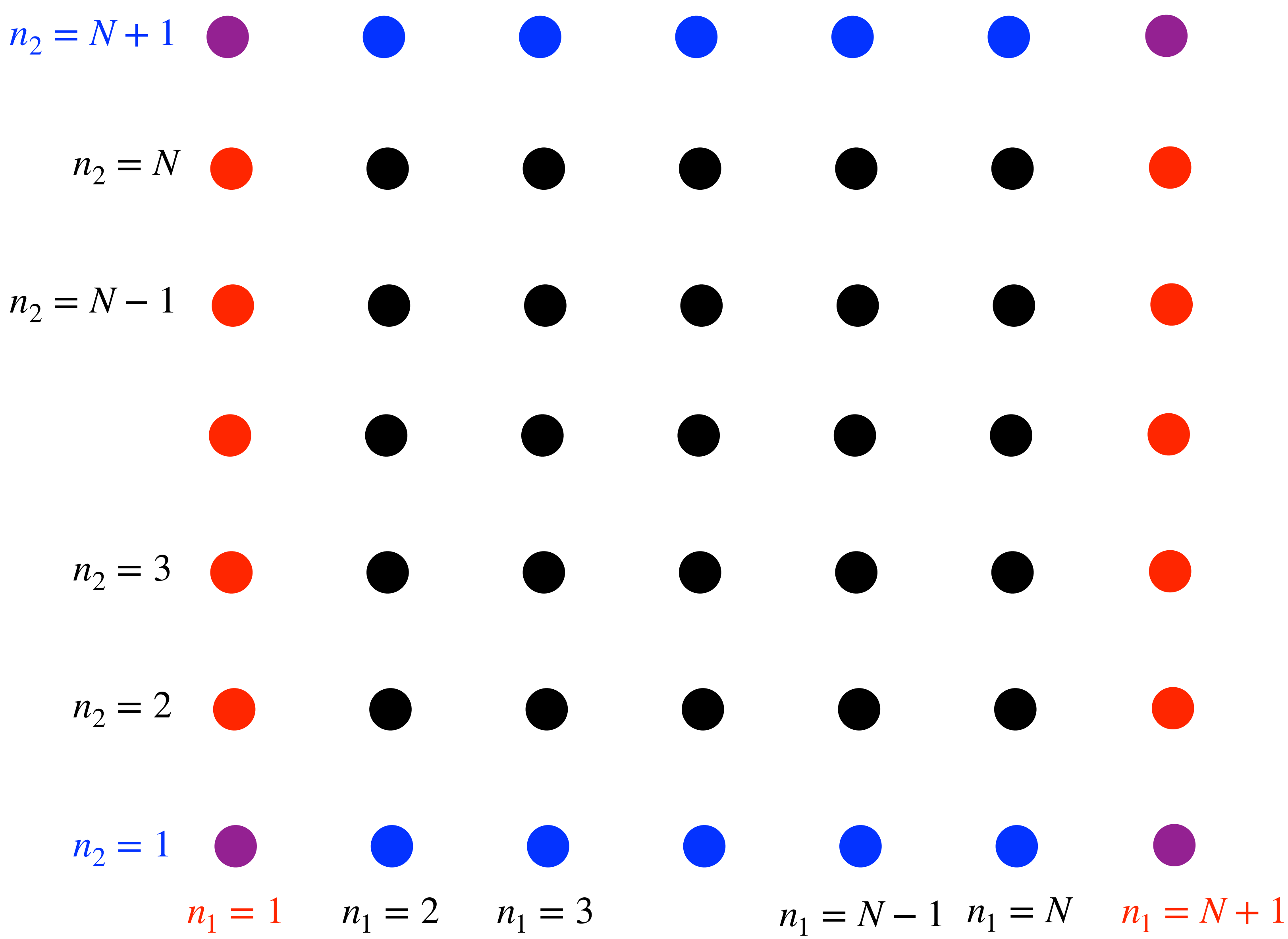
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

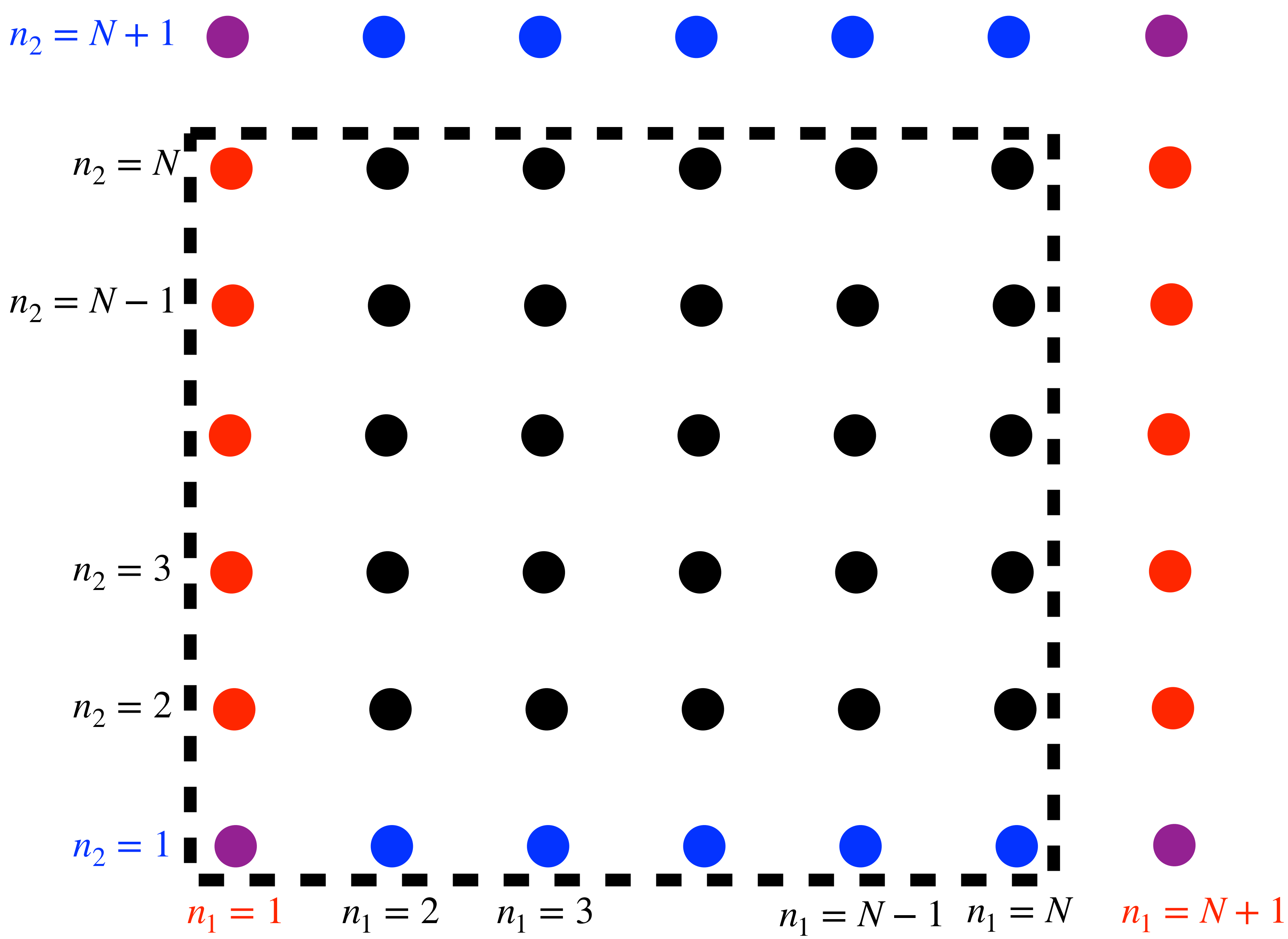
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

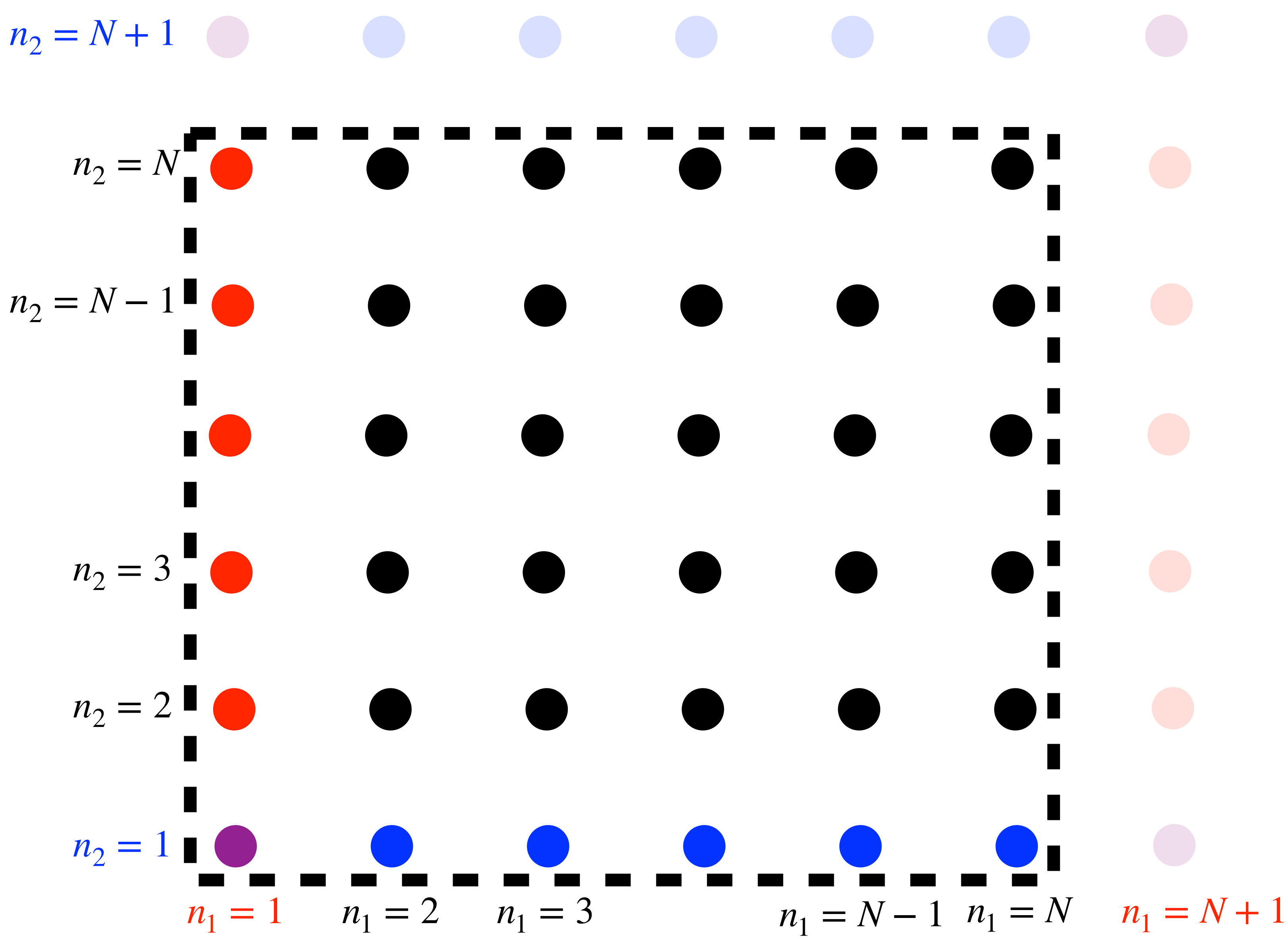
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

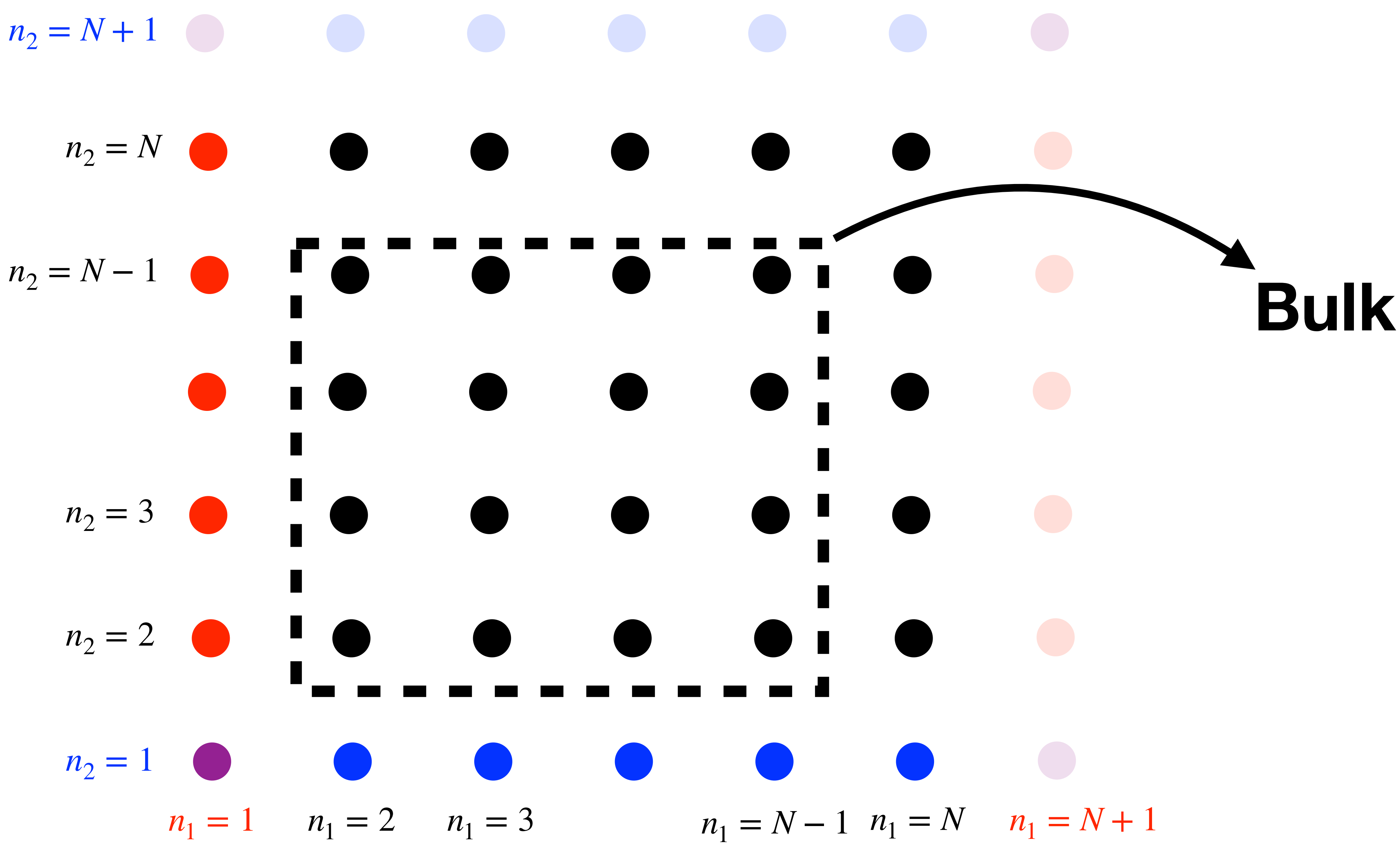
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

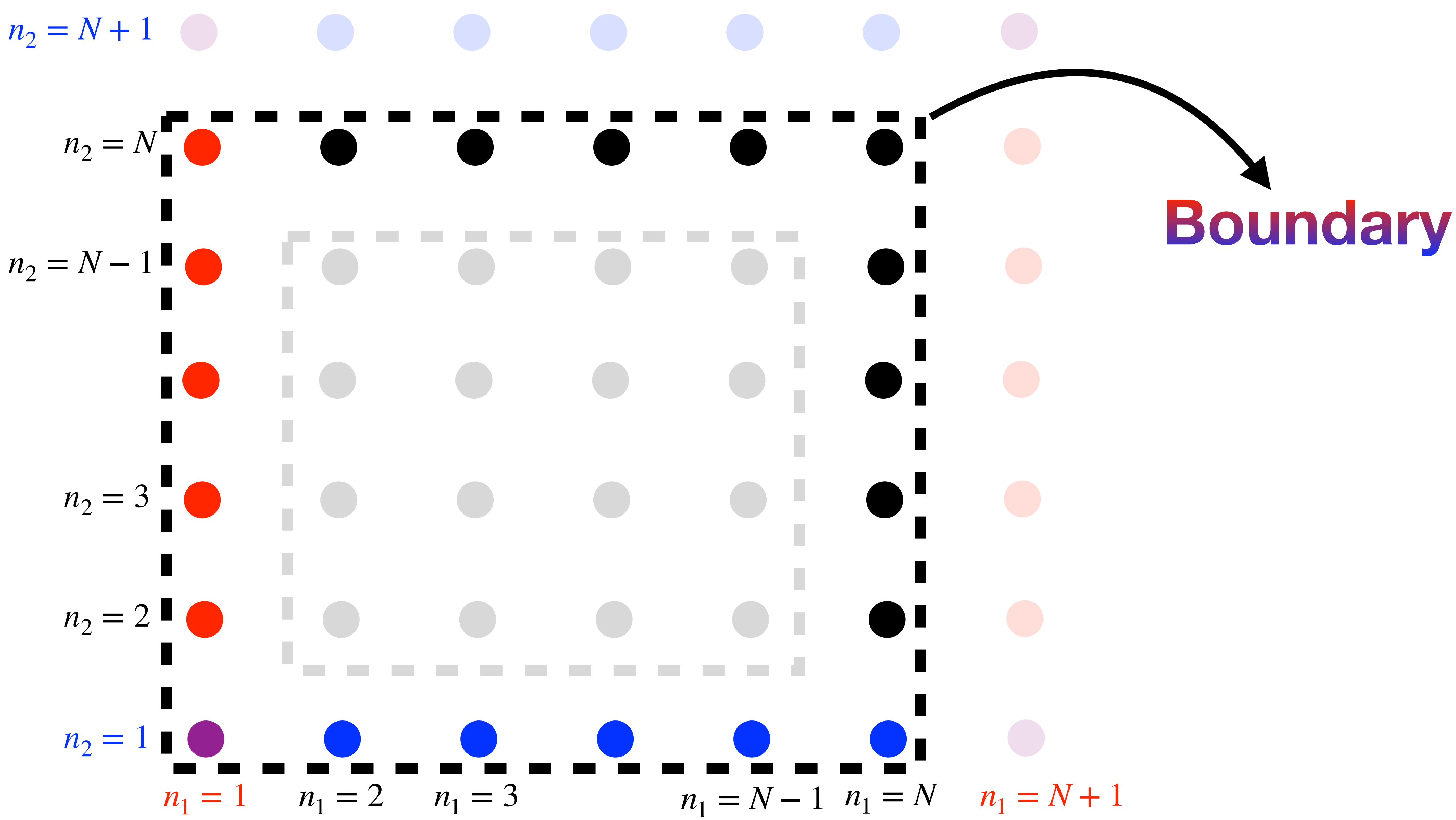
Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 2D



Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 2D

$$\left.\begin{array}{c} \{\mathbf{n} \equiv (n_1, n_2)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\ 2,\ \ldots.,\ N;\ i = 1,2 \\ (N^2 \text{ entries}) \end{array}\right\}$$

Lattice 2D

Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 2D

$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\, 2,\, \ldots.,\, N;\ \ i = 1,2 \\ (N^2 \text{ entries})\end{array}\right\}$$

Periodic ⟺ **Lattice**

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction** (i = 1,2)

Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 2D

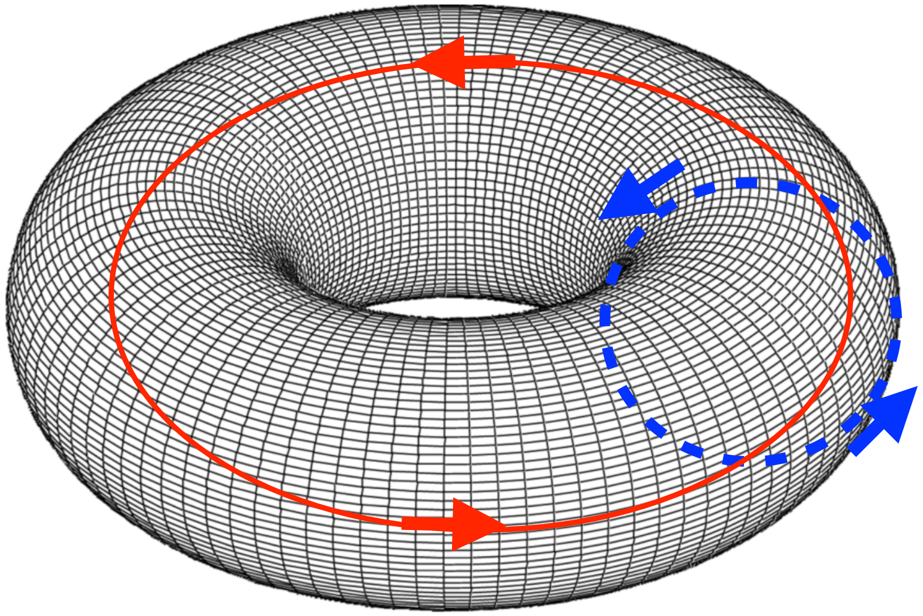
$$\{\mathbf{n} \equiv (n_1, n_2)\}\ ,\ \{f(\mathbf{n})\}\ ,$$
$$n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2$$
$$(N^2 \text{ entries})$$

Periodic ⟺ **Lattice**

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\hat{i}$ ≡ **unit vector in *i*-direction**
(i = 1,2)

Torus (T2)



Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 3D

$$\left.\begin{matrix}\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \;,\; \{f(\mathbf{n})\}\,, \\ n_i = 1,\, 2,\, \ldots\, .\, ,\, N;\;\; i = 1,2,3 \\ (N^3 \text{ entries})\end{matrix}\right\}$$

Lattice 3D

Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 3D

$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\ 2,\ \ldots.,\ N;\ i = 1,2,3 \\ (N^3 \text{ entries})\end{array}\right\}$$

Periodic ⟺ **Lattice**

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)

Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 3D

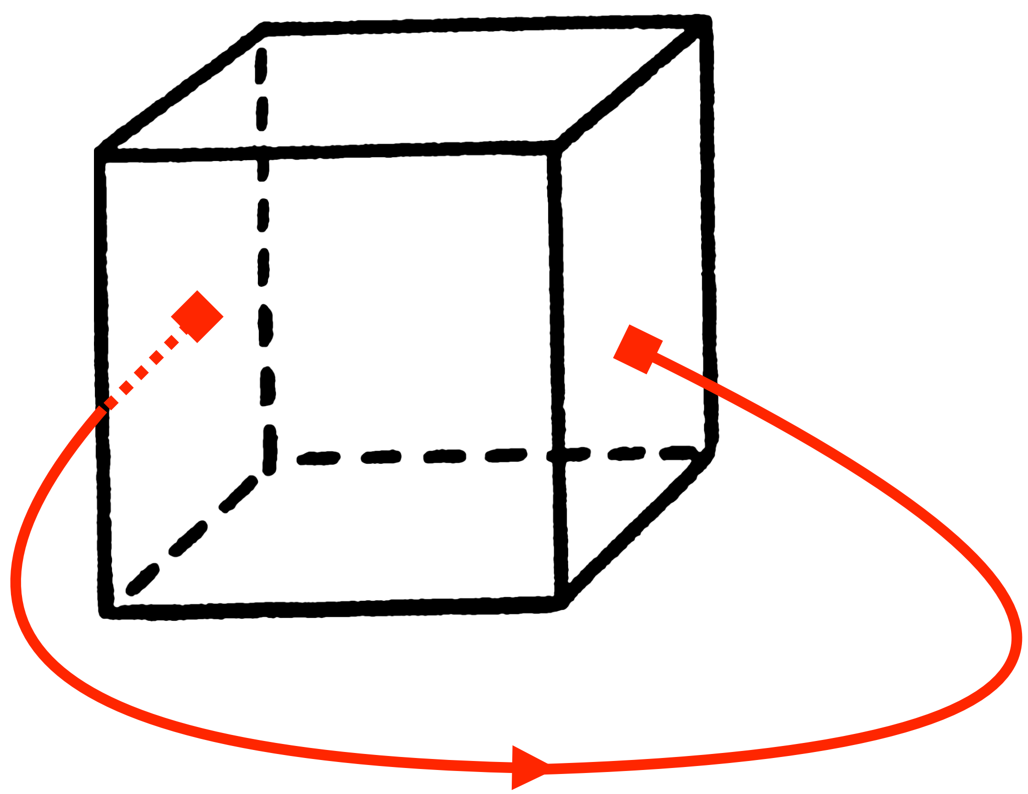
$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f(\mathbf{n})\}\ ,\\ n_i = 1,\, 2,\, \ldots.,\, N;\ i = 1{,}2{,}3 \\ (N^3 \text{ entries})\end{array}\right\}$$

Periodic ⟺ Lattice

$$\boxed{f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)

Torus (T3)

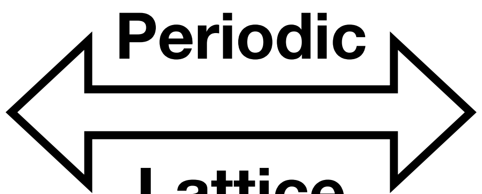


Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 3D

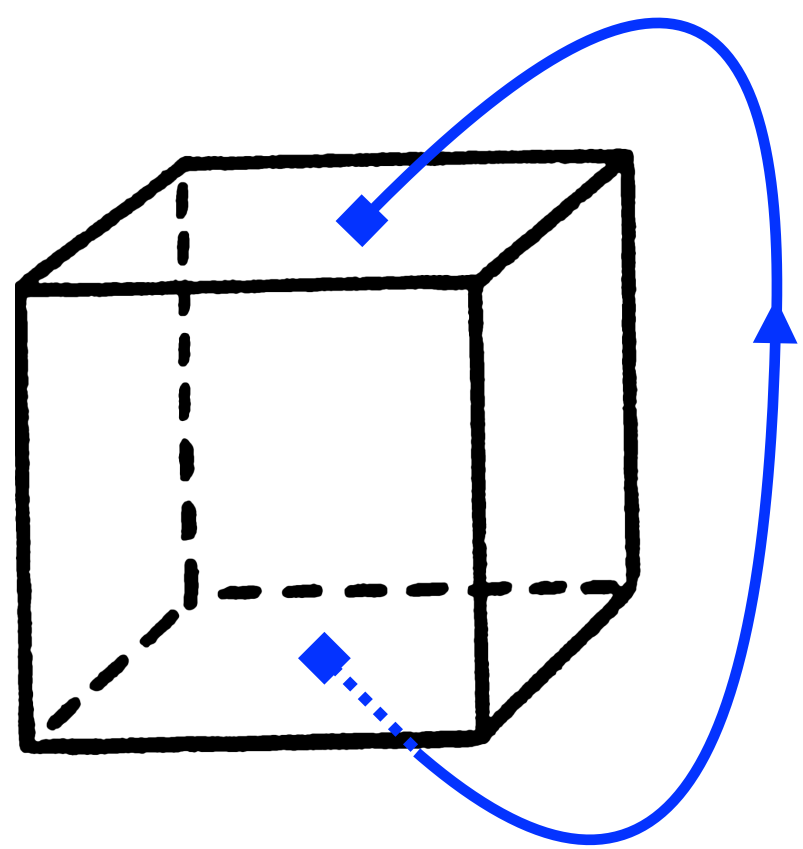
$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \ , \ \{f(\mathbf{n})\} \ , \\ n_i = 1,\ 2,\ \ldots\ldots,\ N;\ \ i = 1,2,3 \\ (N^3 \ \text{entries})\end{array}\right\}$$



$$\boxed{f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in *i*-direction** (i = 1,2,3)

Torus (T3)



Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: 3D

$$\{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f(\mathbf{n})\}\ ,$$

$$n_i = 1,\, 2,\, \ldots,\, N;\;\; i = 1,2,3$$

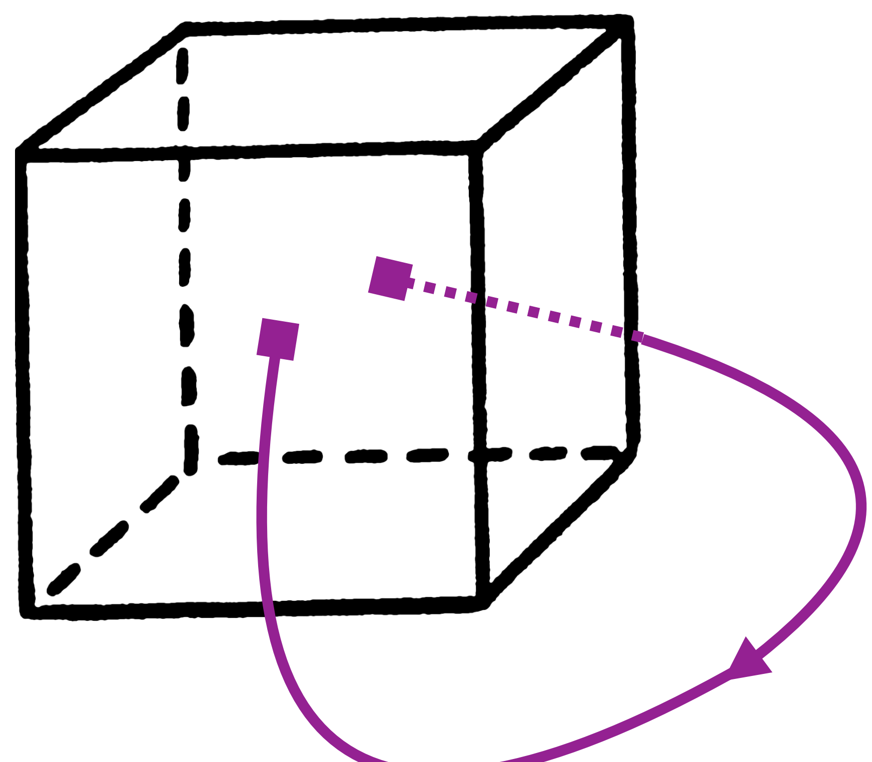
$$(N^3 \text{ entries})$$

Periodic ⟺ **Lattice**

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)

Torus (T3)



Primer on Lattice Techniques

What about the boundaries ?

Periodic Boundary Conditions: d-D

$$\{\mathbf{n} \equiv (n_1, n_2, \ldots, n_d)\} \;,\; \{f(\mathbf{n})\} \;,$$

$$n_i = 1,\, 2,\, \ldots.,\, N\,;\;\; i = 1,2,\ldots,d$$

$$(N^d \text{ entries})$$

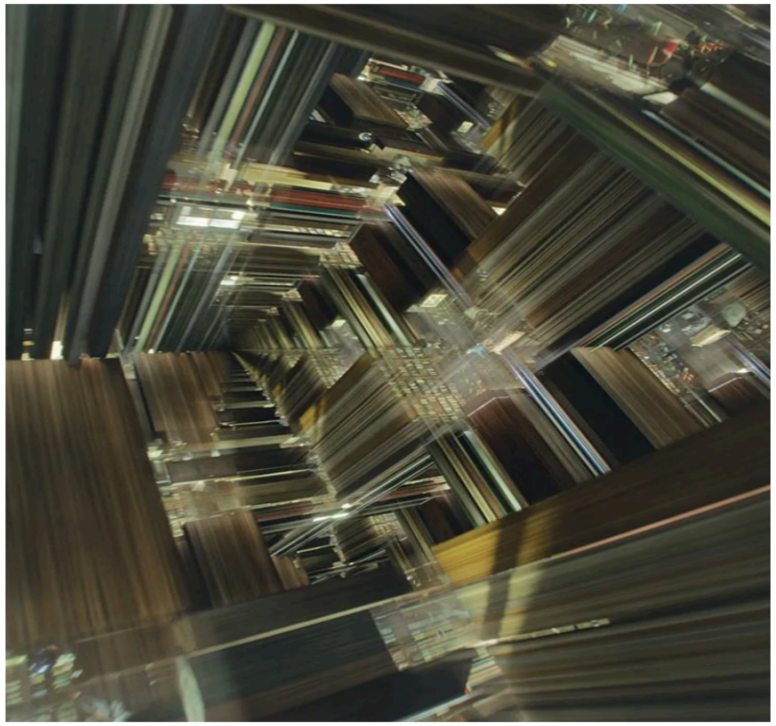
Periodic ⟺ **Lattice**

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**

(i = 1,2,…,d)

Torus (Td)



Primer on Lattice Techniques

Definition of a Lattice (3D)

$$\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \ , \ \{f_j(\mathbf{n})\}$$

$$n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2,3$$

$(N^3 \text{ sites})$ [j = 1, 2 ,…, #]

Periodic ⟺ **Lattice**

$$f_j(\mathbf{n} + N\hat{i}) \equiv f_j(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)

Primer on Lattice Techniques

Definition of a Lattice (3D)

$$\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \;,\; \{f_j(\mathbf{n})\}$$

$$n_i = 0,\ 1,\ \ldots\ldots,\ N-1\ ;\ \ i = 1,2,3$$

$(N^3 \text{ sites})$ [j = 1, 2 ,..., #]

Periodic ⟺ Lattice

$$f_j(\mathbf{n} + N\hat{i}) \equiv f_j(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**

(i = 1,2,3)

Primer on Lattice Techniques

Definition of a Lattice (3D)

$$\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \;,\; \{f_j(\mathbf{n})\}$$

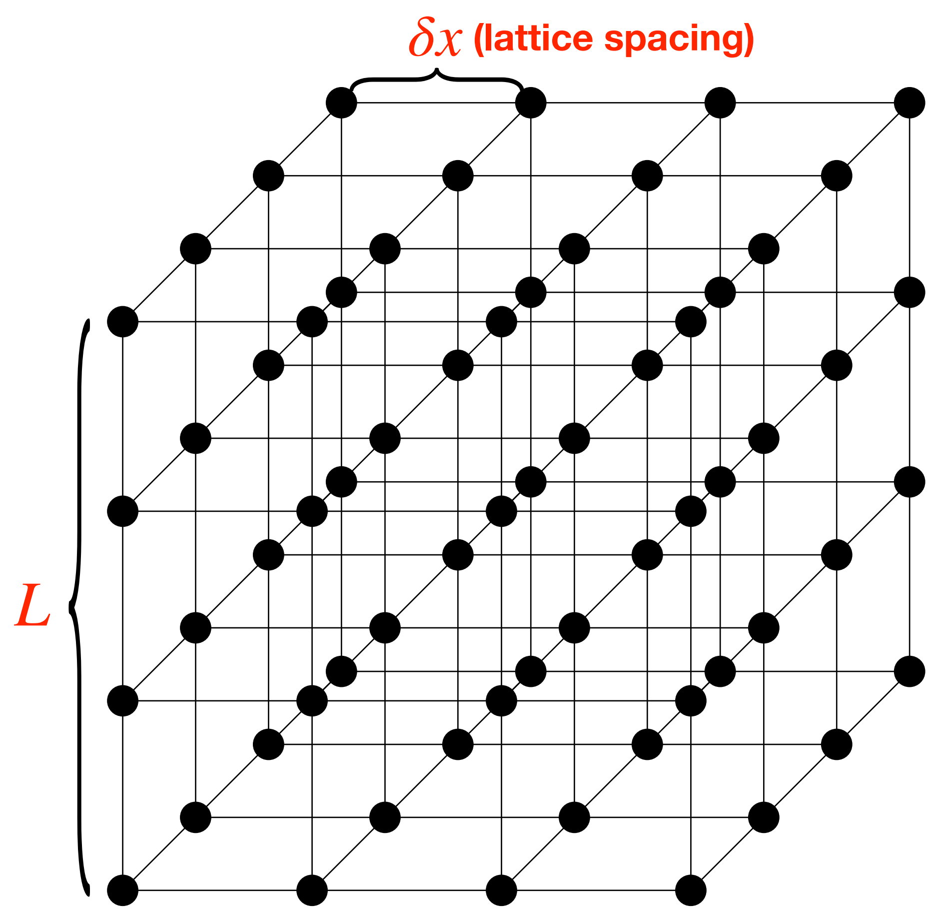
$$n_i = 0,\ 1,\ \ldots.,\ N-1\ ;\ \ i = 1,2,3$$

$(N^3$ sites) [j = 1, 2 ,..., #]

Periodic ⇔ **Lattice**

$$f_j(\mathbf{n} + N\hat{i}) \equiv f_j(\mathbf{n})$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)



- N : number of points/dimension
- $L = N \cdot \delta x$: length side
- $\delta x \equiv \dfrac{L}{N}$: lattice spacing

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

$$\left(\sum_{\mathbf{n}} e^{i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} = N^3 \delta_{\mathbf{0},\tilde{\mathbf{n}}} \right)$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$\left. \begin{array}{c} f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n}) \\ (i = 1,2,3) \end{array} \right\}$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$(i = 1,2,3)$

$\Rightarrow$

Discrete Momentum: $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$

$$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2}$$

$(i = 1,2,3)$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$

$n_i = 1,2,\ldots,N$

$\Rightarrow$

Periodic FT modes: $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$

$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}})$, $i = 1,2,3$

$\hat{i} \equiv$ unit vector in $\tilde{n}_i$-direction

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1,2,\ldots,N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

$\Rightarrow$

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1,2,\ldots,N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

Real Lattice

⇒

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Reciprocal Lattice

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$n_i = 1,2,\ldots,N$

$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$

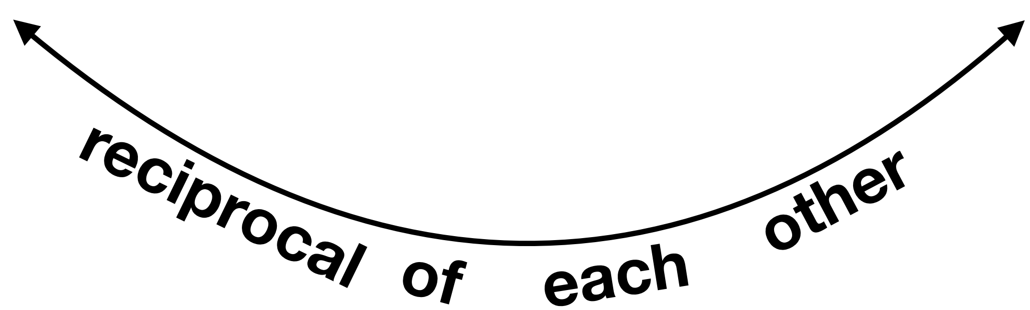
Real Lattice

Momentum Lattice Periodic

$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2}$

$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$

Fourier Lattice



Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3}\sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1,2,\ldots,N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

Real Lattice

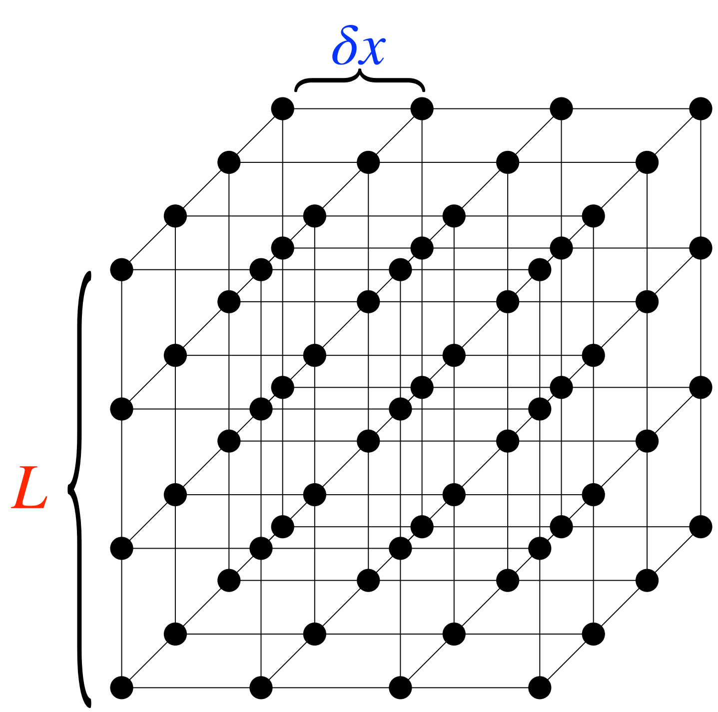
⟺

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1,\ldots,\ -1,\ 0,\ 1,\ldots,\ \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Fourier Lattice



N : # points/dimension

$L = N \cdot \delta x$: length side

$\delta x \equiv \frac{L}{N}$: lattice spacing

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3}\sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1,2,\ldots,N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

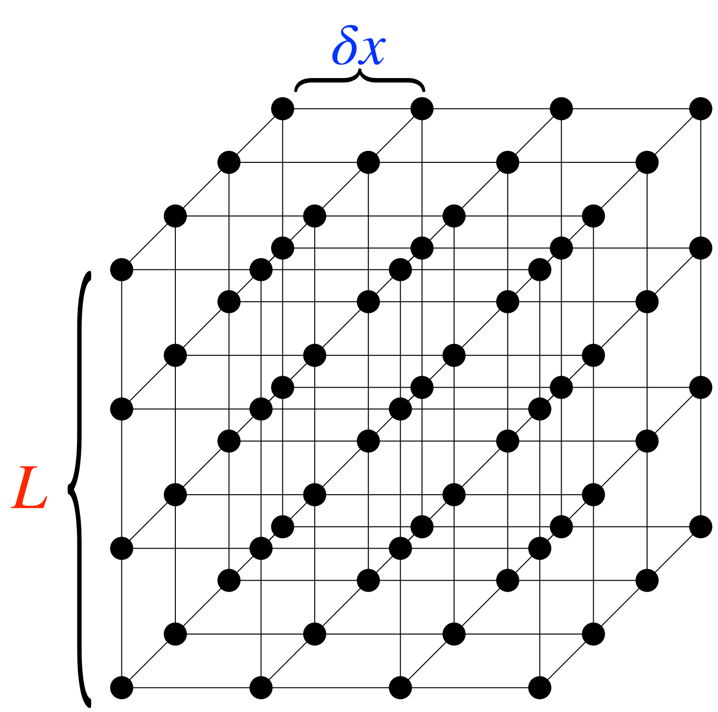
Real Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1,\ldots,\; -1,\; 0,\; 1,\ldots,\; \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Fourier Lattice



N : # points/dimension

$L = N\cdot\delta x$: length side

$\delta x \equiv \frac{L}{N}$: lattice spacing

Minimum and maximum momenta:

$$k_{\rm min} = \frac{2\pi}{L} \qquad k_{\rm max}^{(i)} = \frac{1}{2}Nk_{\rm min}$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1, 2, \ldots, N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

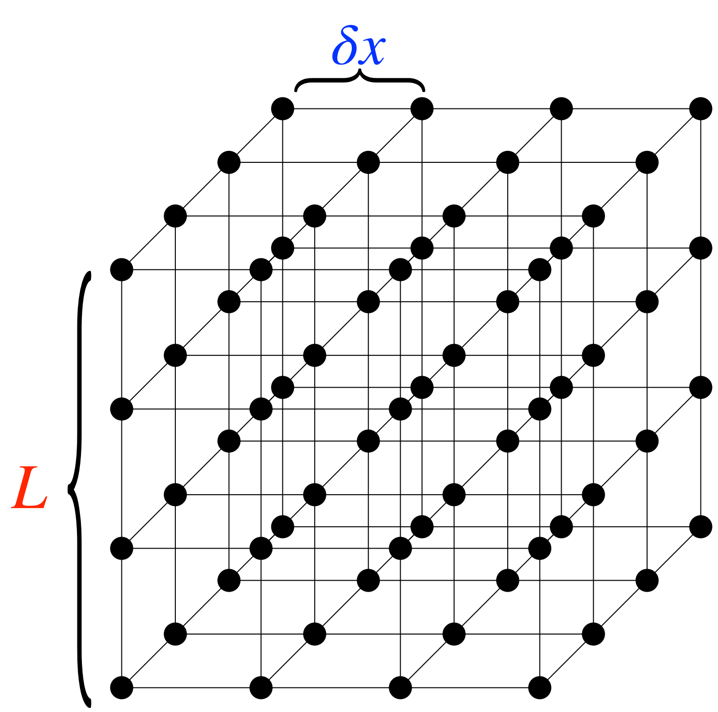
$\Leftrightarrow$

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Real Lattice



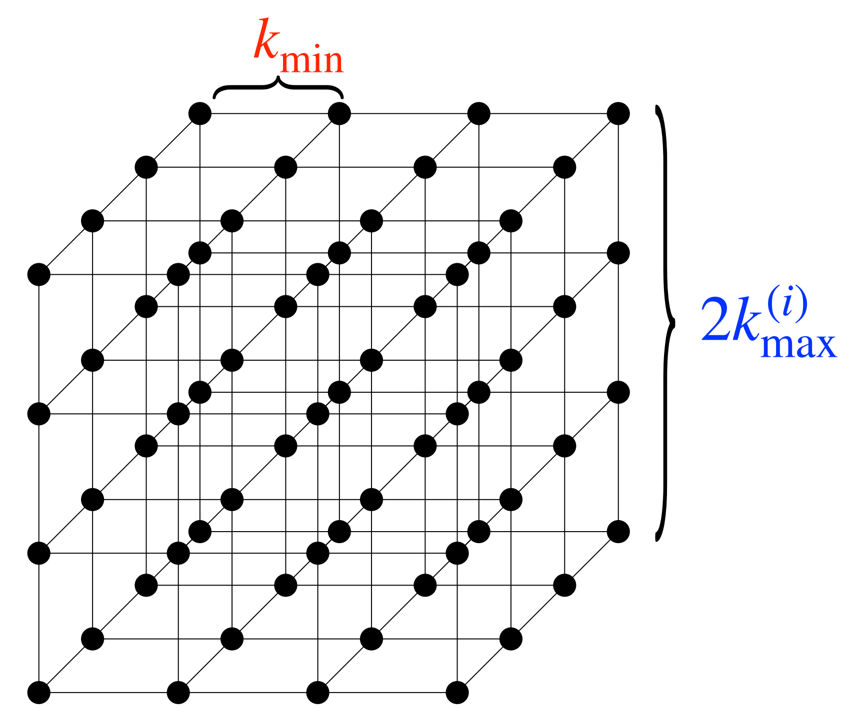
Fourier Lattice

N : # points/dimension

$L = N \cdot \delta x$: length side

$\delta x \equiv \dfrac{L}{N}$: lattice spacing

$\Rightarrow$



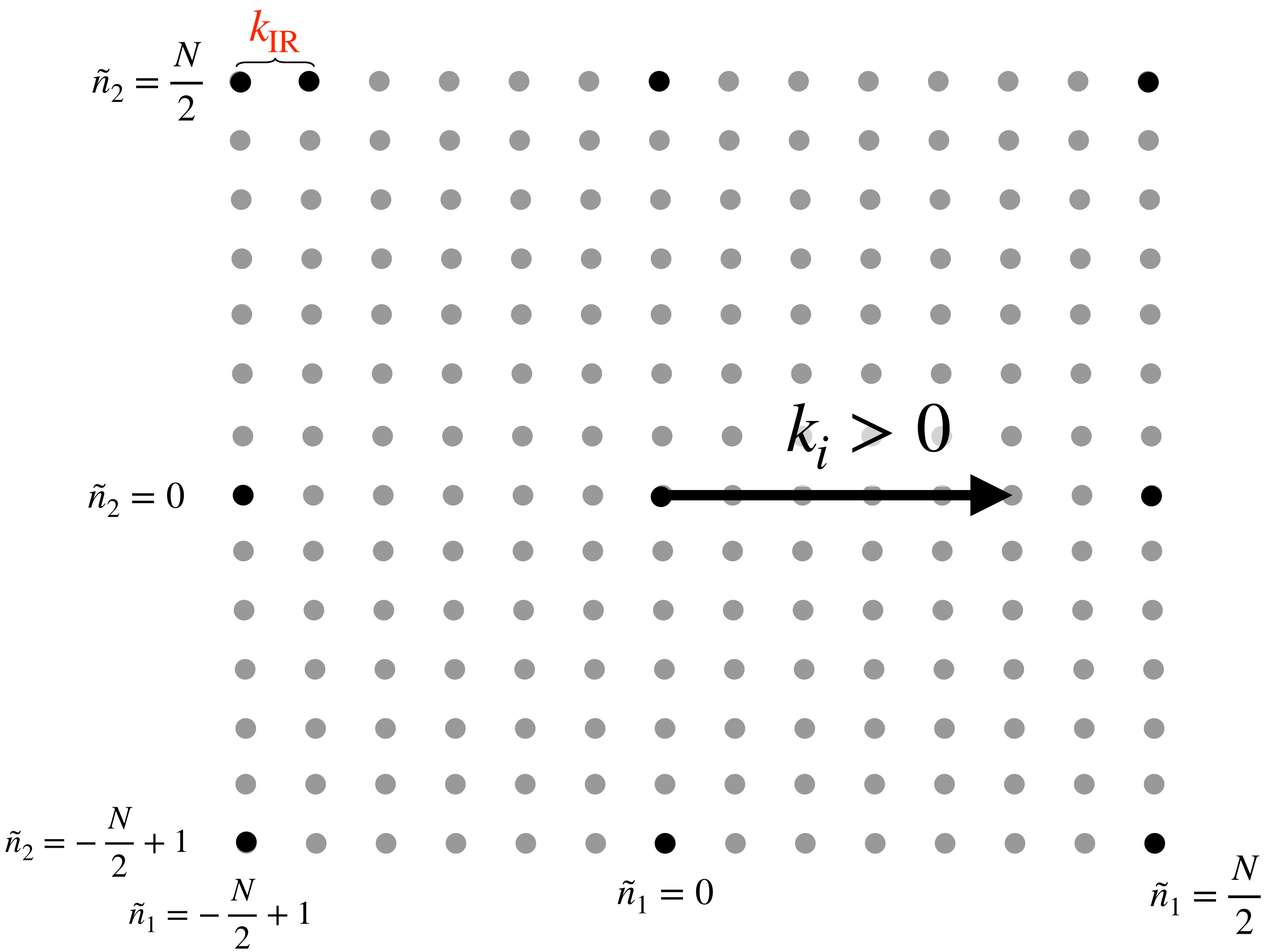
Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d)$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

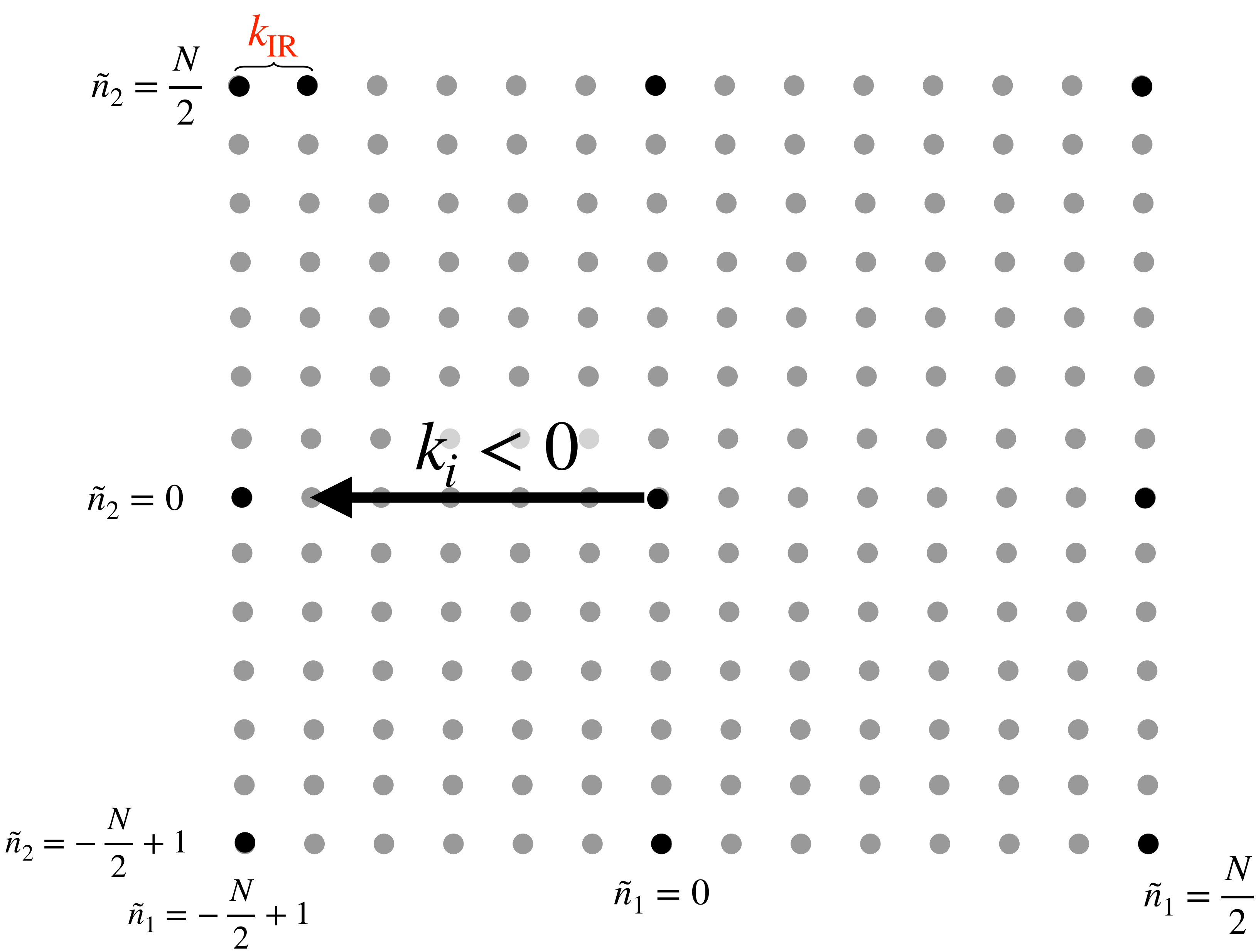
Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1, \ldots, \; -1,\; 0,\; 1, \ldots, \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d)$$



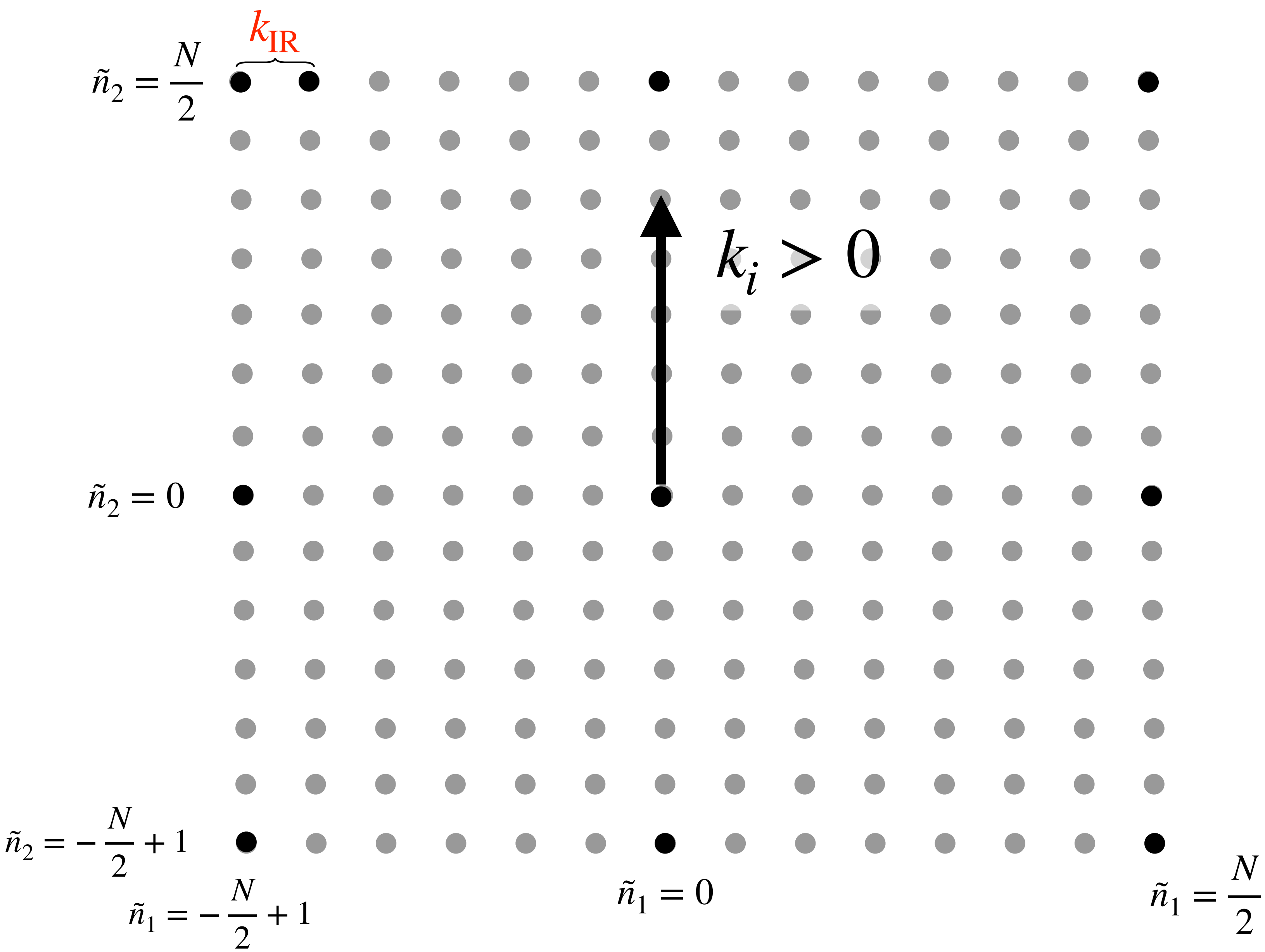
$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1,\ldots,\ -1,\ 0,\ 1,\ldots,\ \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d) \end{array} \right\}$$



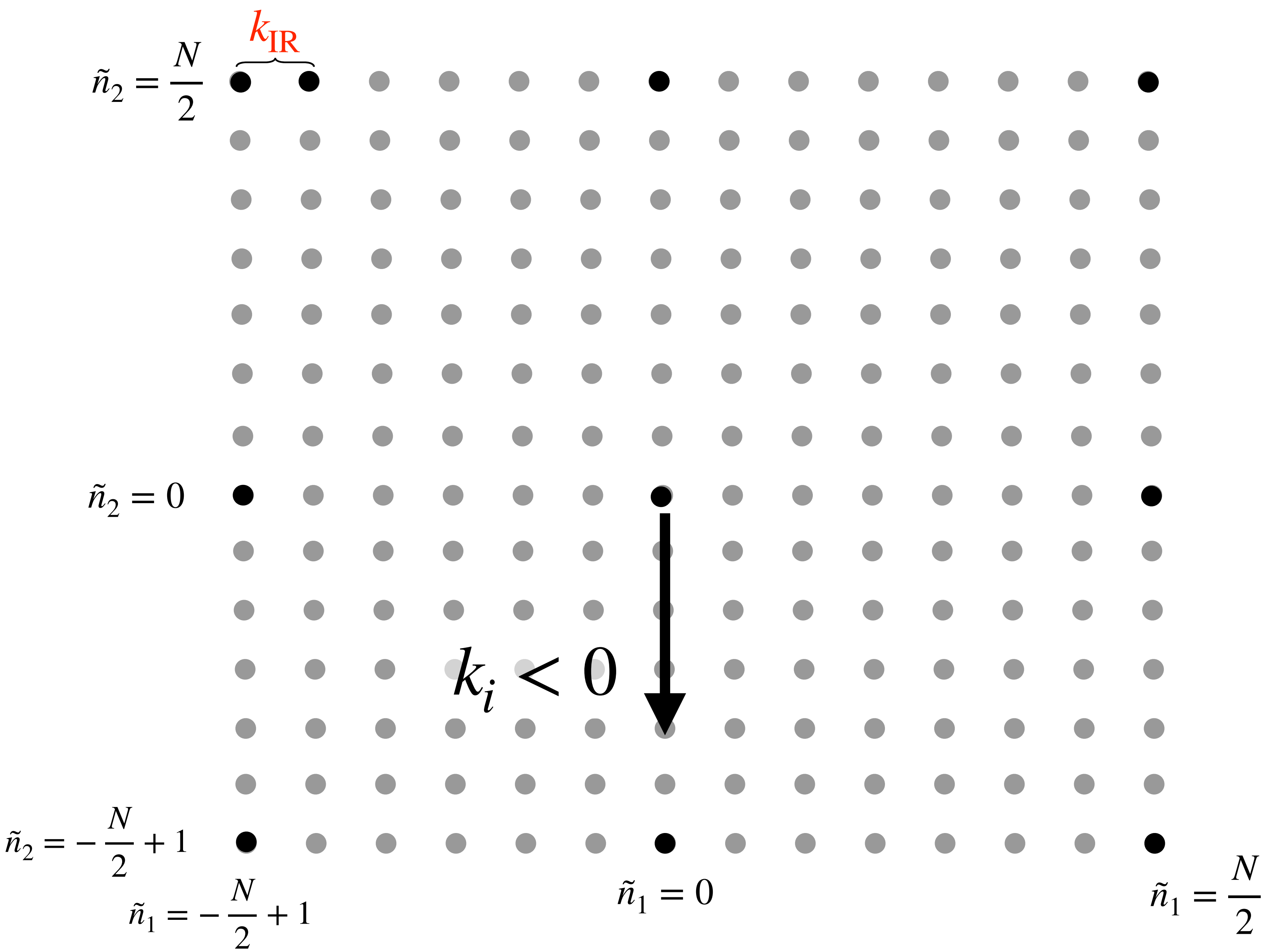
$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, \; -1,\; 0,\; 1, \ldots, \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d) \end{array} \right\}$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

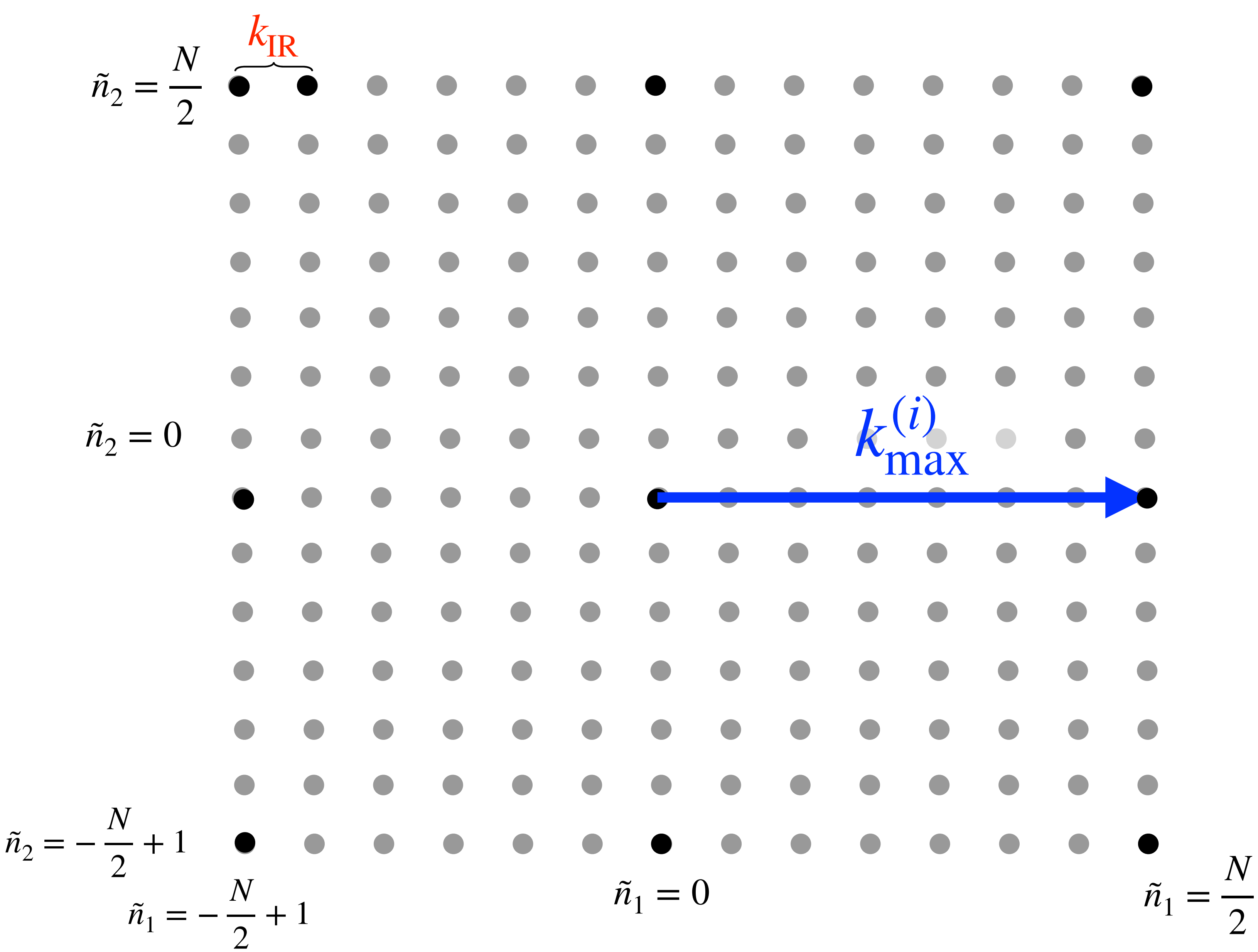
Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1, \ldots, \; -1,\; 0,\; 1, \ldots, \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d)$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

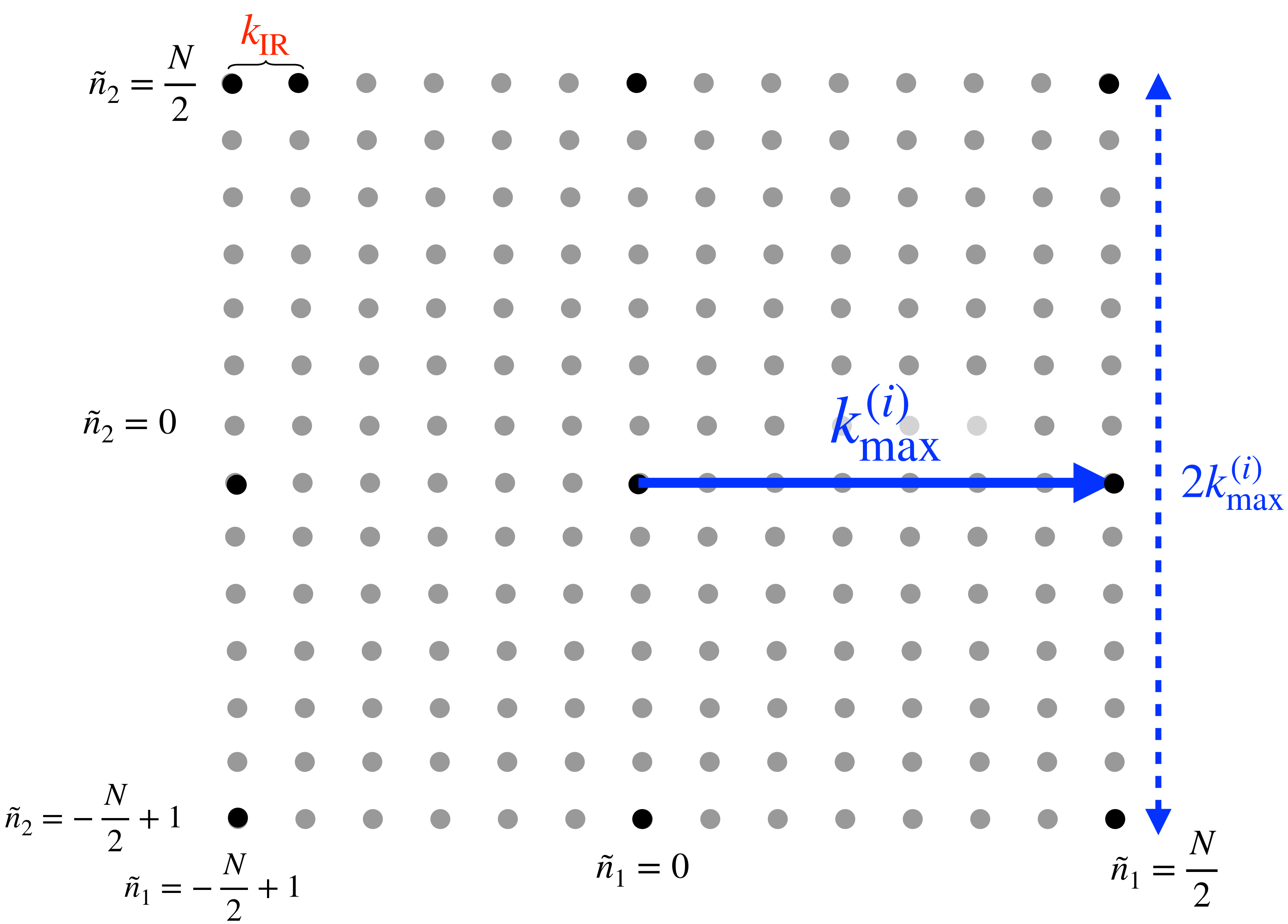
$$k^{(i)}_{\rm max} \equiv \frac{N}{2} k_{\rm IR}$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, -1,\, 0,\, 1, \ldots, \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d) \end{array} \right\}$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

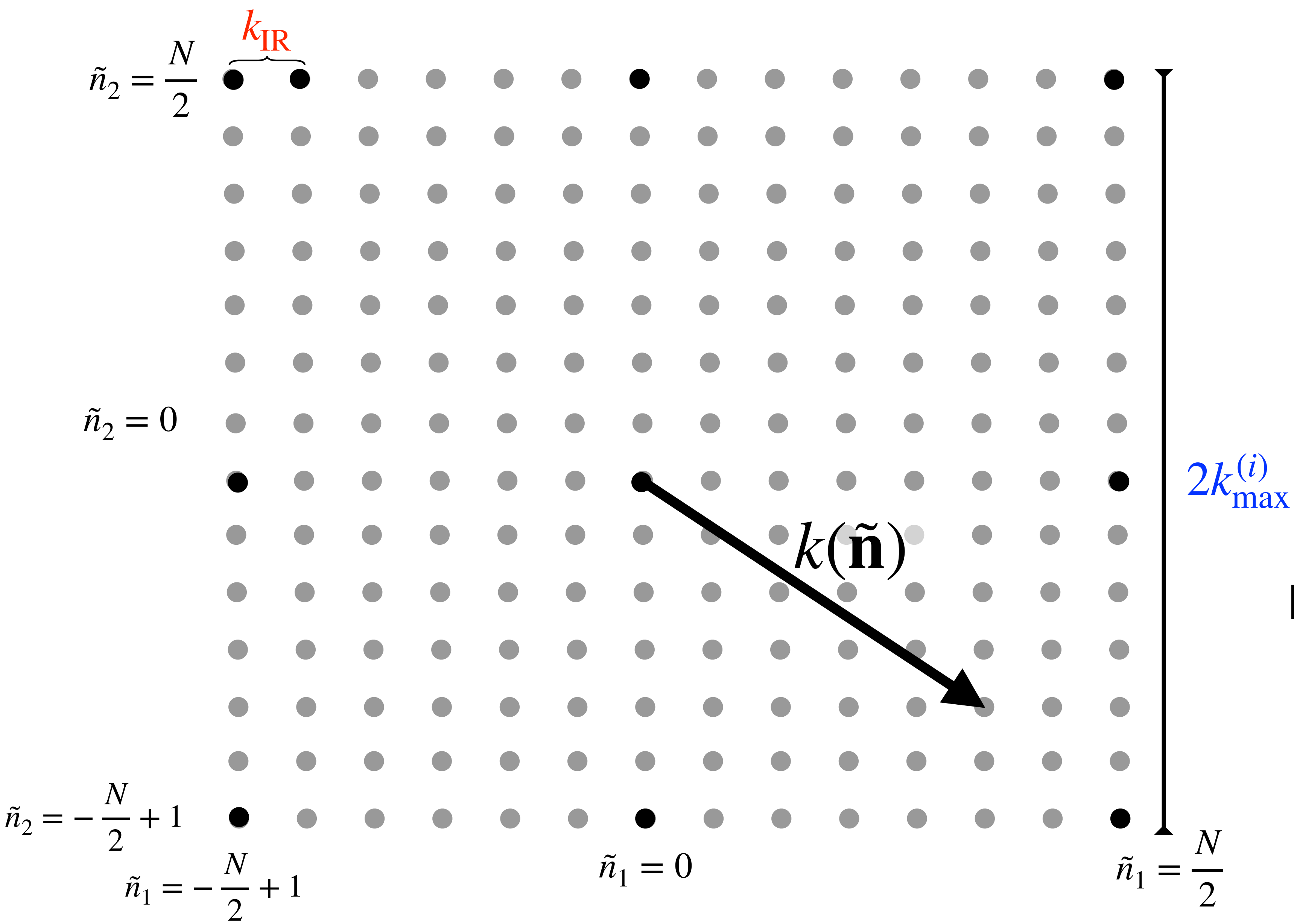
$$k^{(i)}_{\rm max} \equiv \frac{N}{2} k_{\rm IR}$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d) \end{array} \right\}$$



$$k_{IR} \equiv \frac{2\pi}{L}$$

$$k_{max}^{(i)} \equiv \frac{N}{2} k_{IR}$$

Linear Momentum

$$k(\tilde{\mathbf{n}}) \equiv \tilde{\mathbf{n}} k_{IR}$$

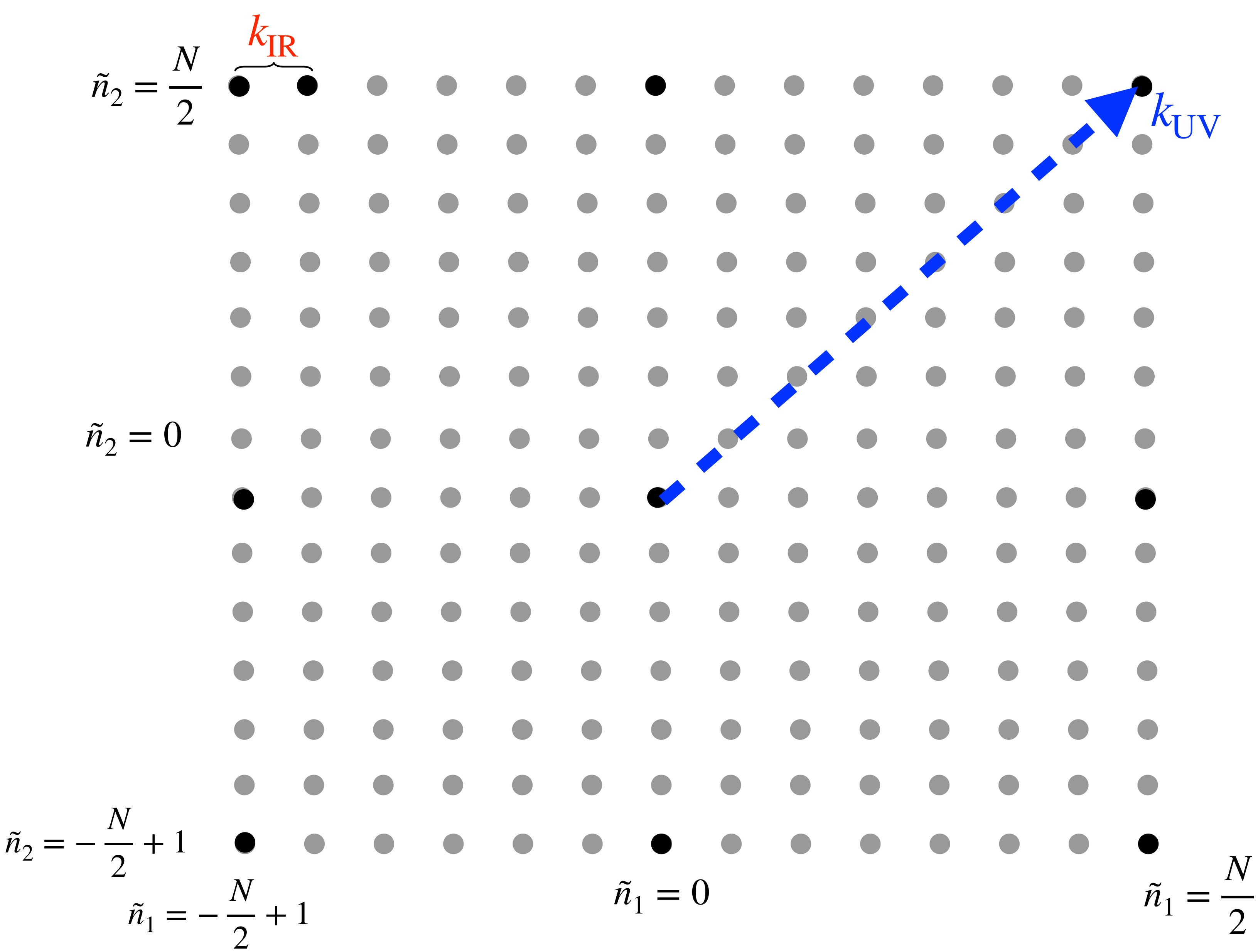
$$\tilde{\mathbf{n}} \equiv (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\ldots,d) \end{array} \right\}$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3}\sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$$n_i = 1,2,\ldots,N$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

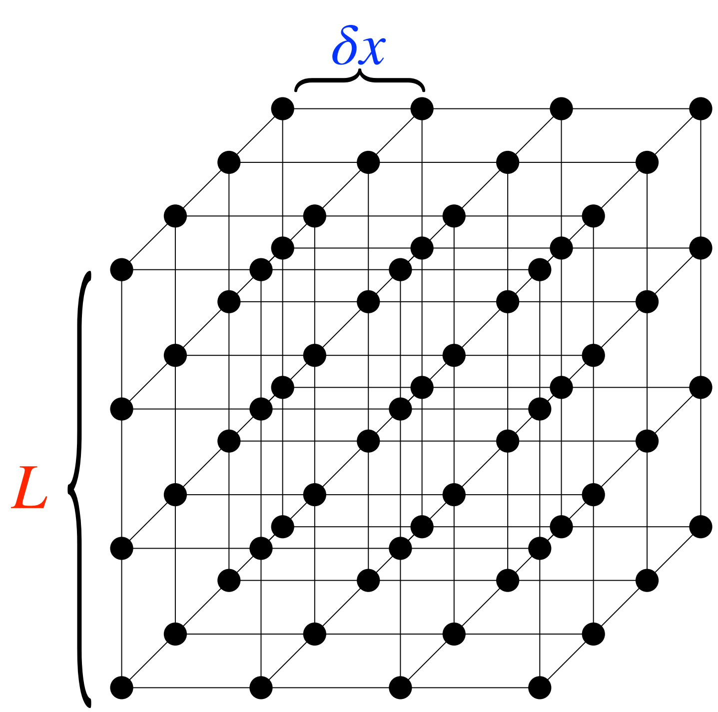
Real Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1,\ldots, \; -1,\; 0,\; 1,\ldots,\; \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$$

Fourier Lattice



N : # points/dimension

$L = N \cdot \delta x$: length side

$\delta x \equiv \frac{L}{N}$: lattice spacing

IR (min) and UV (max) momenta:

$$k_{\rm IR} = \frac{2\pi}{L} \qquad k_{\rm UV} = \frac{\sqrt{3}}{2} N k_{\rm min}$$

Primer on Lattice Techniques

Definition of a Fourier Transform

$$f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}}) \quad \Leftrightarrow \quad f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n})$$

Periodic Lattice

$n_i = 1,2,\ldots,N$

$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$

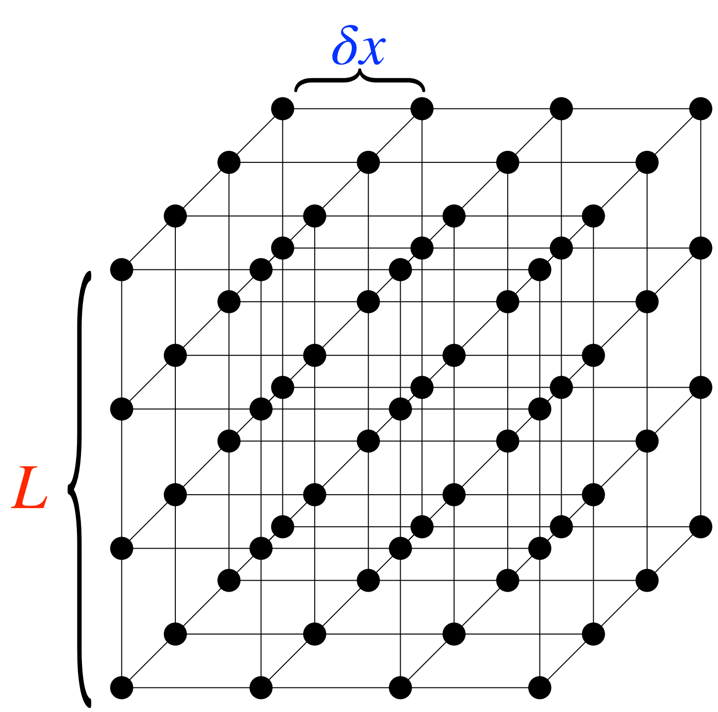
Momentum Lattice Periodic

$\tilde{n}_i = -\frac{N}{2} + 1, \ldots, \; -1,\; 0,\; 1, \ldots, \; \frac{N}{2}$

$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,3)$

Real Lattice

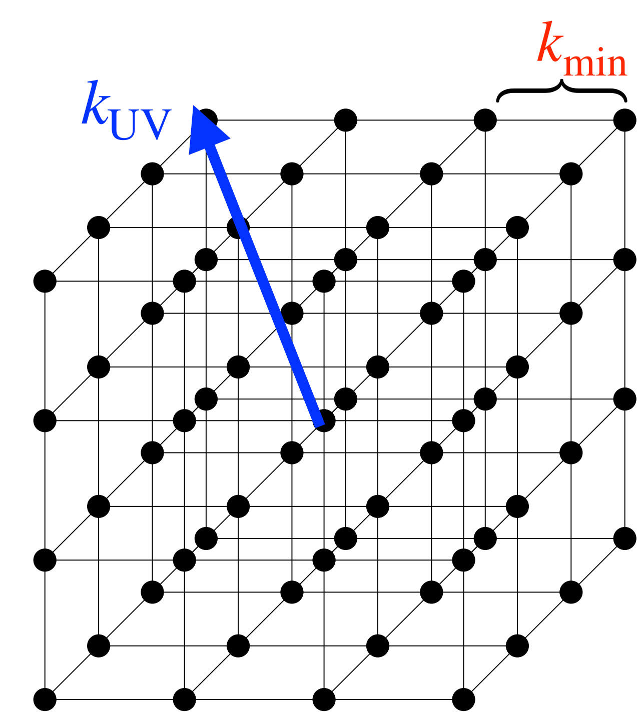
Fourier Lattice



N : # points/dimension

$L = N \cdot \delta x$: length side

$\delta x \equiv \frac{L}{N}$: lattice spacing



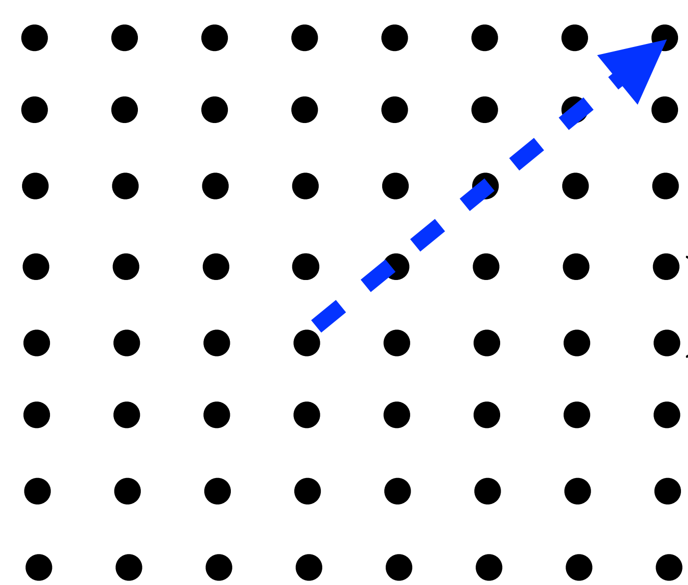
Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1, 2, \ldots, d) \end{array} \right\}$$

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

Primer on Lattice Techniques

Definition of Fourier Lattice

Real Periodic Lattice

$$n_i = 1,2,\ldots,N \quad (i = 1,2,\ldots,d)$$

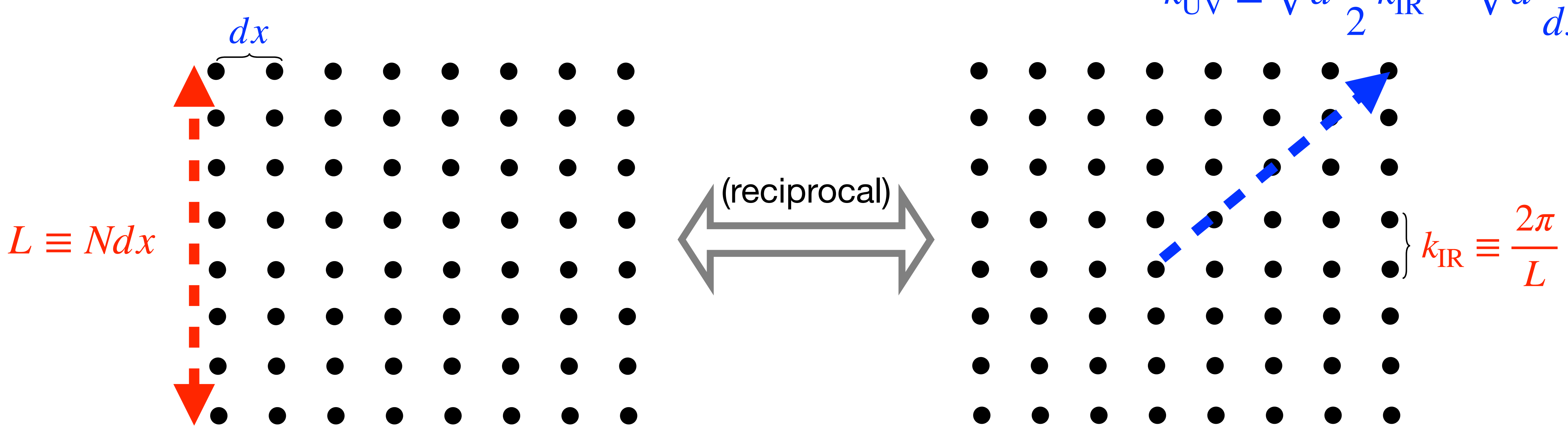
$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1,\ldots, \; -1,\; 0,\; 1,\ldots, \frac{N}{2}$$

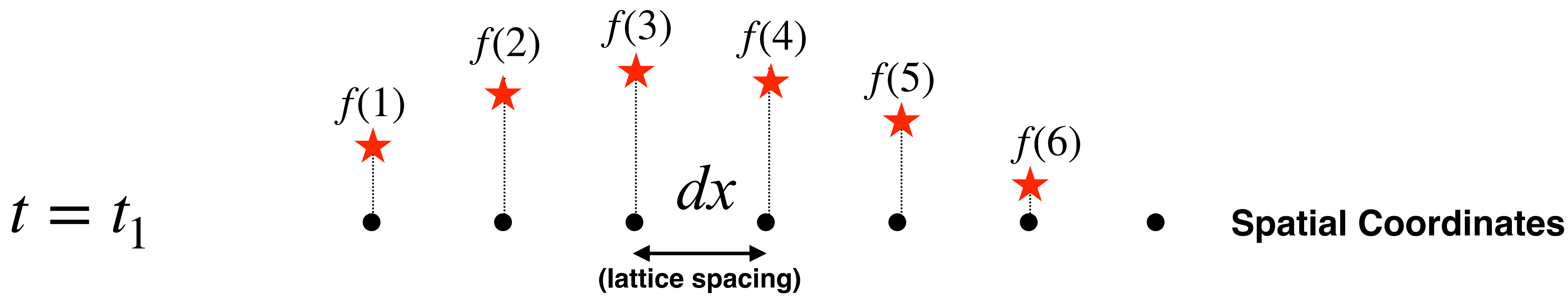
$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,\ldots,d)$$

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$



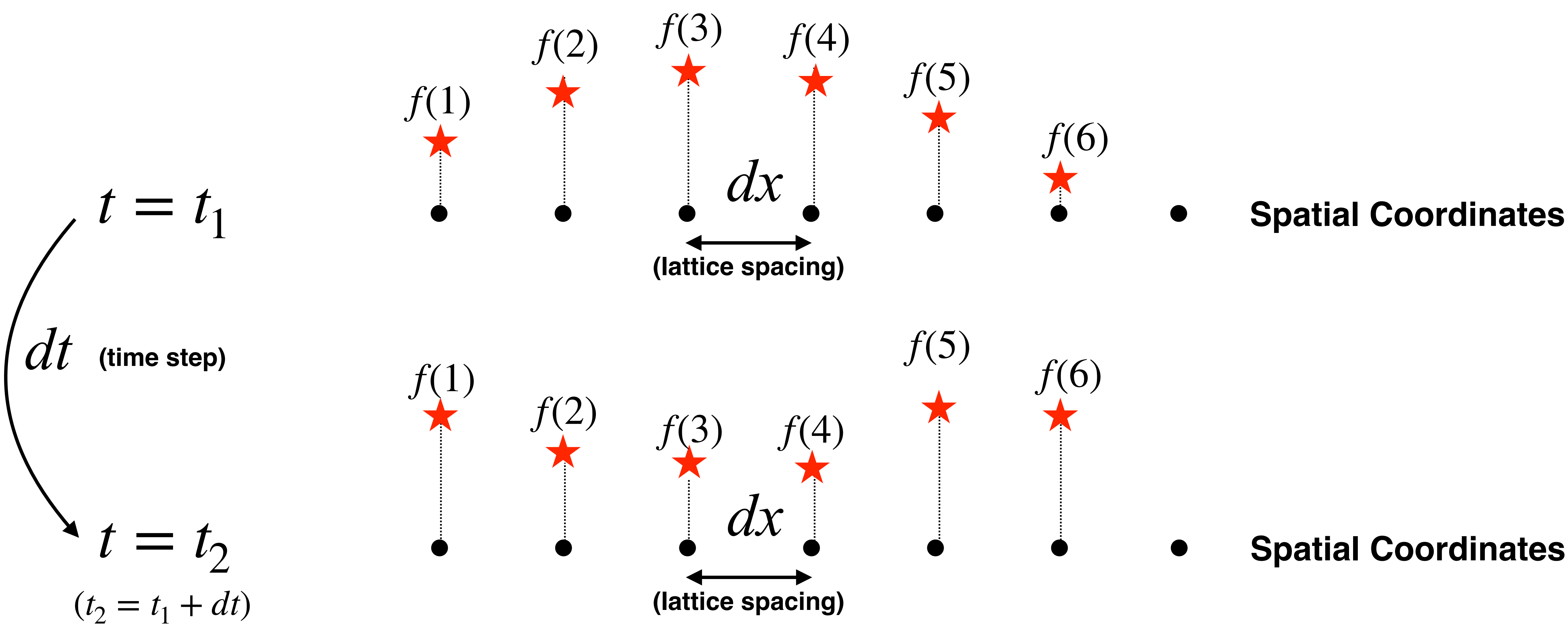
Primer on Lattice Techniques

Lattice Derivatives



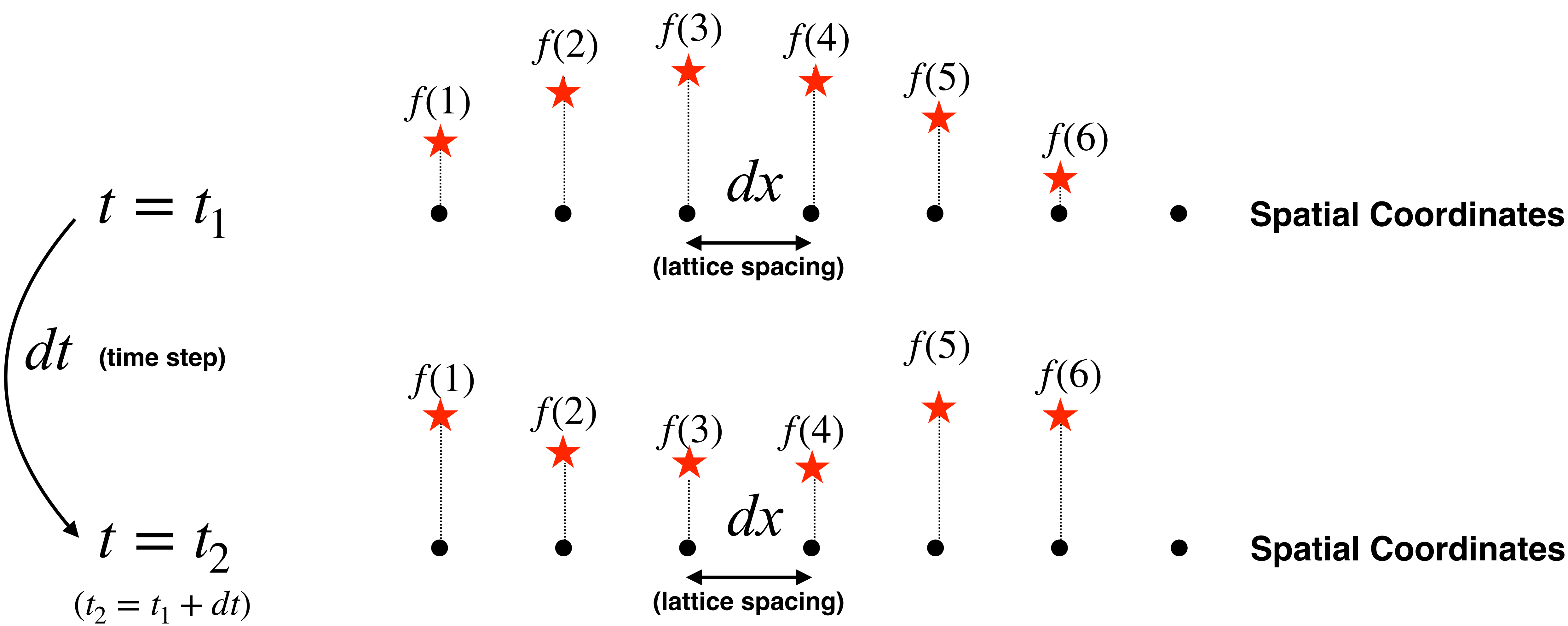
Primer on Lattice Techniques

Lattice Derivatives



Primer on Lattice Techniques

Lattice Derivatives



New notation: $n \equiv (n_0, \mathbf{n}) = (n_0, n_1, n_2, n_3)$

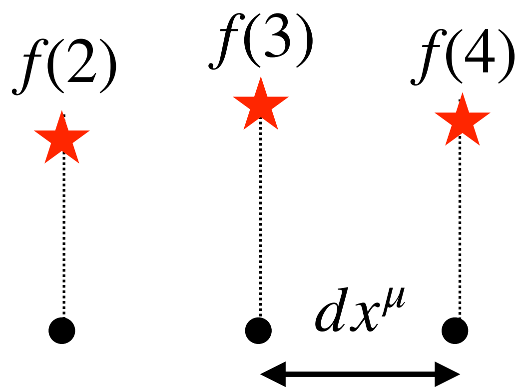
$n_i = 0,\ 1, ...,\ N - 1\ ; i = 1,2,3$

$n_0 = 1,\ 2,\ 3,\ \ldots,\ \#\ \text{(time steps)}$

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu}$$

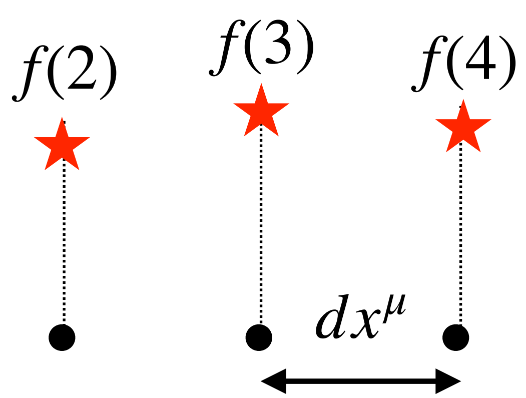


$$[\nabla_\mu^{(0)} f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $[\nabla^{(0)}_\mu f](n) = \dfrac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$



$$[\nabla^{(0)}_\mu f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

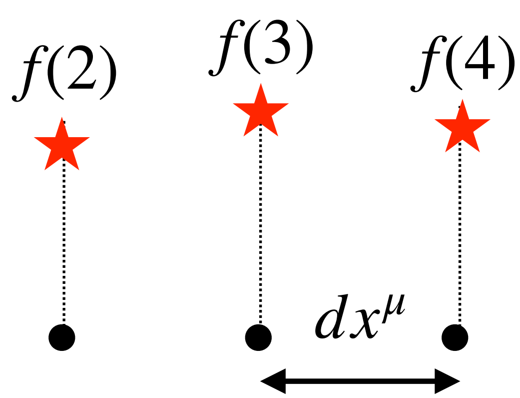
Primer on Lattice Techniques

Lattice Derivatives

Neutral:

$$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$$

$dx,\ dt$



$$[\nabla_\mu^{(0)} f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

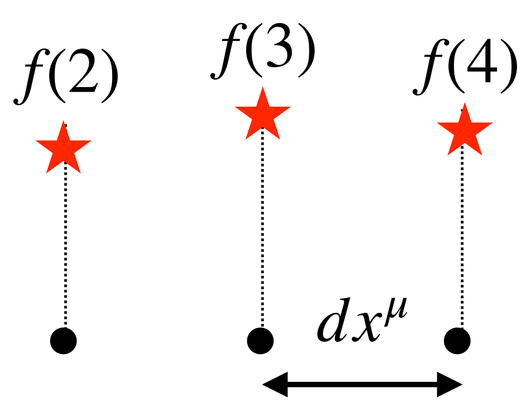
Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu}$$

(+) Forward ; (-) Backward



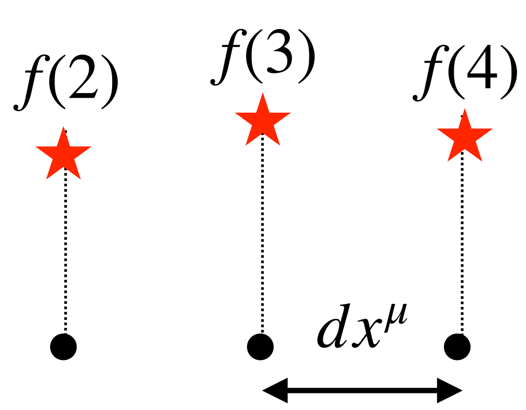
Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\big|_{x\,\equiv\,\mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\big|_{x\,\equiv\,(n\pm\hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward



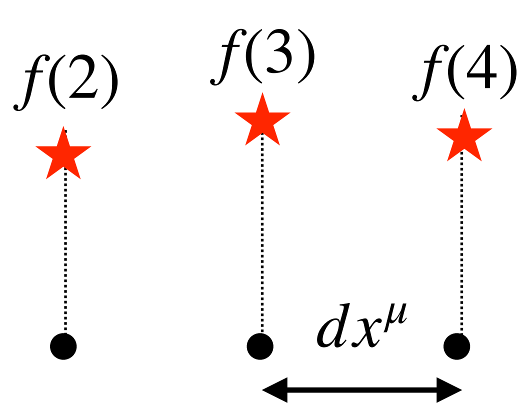
Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n \pm \hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\Big|_{x \equiv (n \pm \hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward



$$[\nabla_\mu^{(+)} f](3.0) \equiv \frac{f(4) - f(3)}{dx^\mu} \longrightarrow \partial_\mu \mathtt{f}(3) + \mathcal{O}(dx^\mu)$$

$$[\nabla_\mu^{(+)} f](3.5) \equiv \frac{f(4) - f(3)}{dx^\mu} \longrightarrow \partial_\mu \mathtt{f}(3.5) + \mathcal{O}(dx_\mu^2)$$

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n \pm \hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu) \\ \\ \partial_i \mathtt{f}(x)\big|_{x \equiv (n \pm \hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward

Arbitrary: $$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m)$$

(Lin. Combination)

weights: $D_i(l,m) \longrightarrow$ real-valued function of two lattice coordinates

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\Big|_{x\equiv (n\pm\hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward

Arbitrary: $$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m)$$

(invariant under Translations)

(Lin. Combination)

weights: $D_i(l,m) \longrightarrow$ real-valued function of two lattice coordinates

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla^{(0)}_\mu f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0 dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla^\pm_\mu f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0 dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\Big|_{x\equiv (n\pm\hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward

Arbitrary: $$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m) \overset{\text{shift}}{=} \sum_{m'} D_i(m') f(l-m')$$

(Lin. Combination)

(invariant under Translations)

weights: $D_i(l,m) \longrightarrow$ real-valued function of two lattice coordinates

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\big|_{x\equiv (n\pm\hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward

Arbitrary: $$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m) = \sum_{m'} D_i(m') f(l-m')$$

(specially useful for Laplacians)

Primer on Lattice Techniques

Lattice Derivatives

Neutral: $$[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0 dt} + \mathcal{O}(dx_\mu^2)$$

Charged: $$[\nabla_\mu^{\pm} f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0 dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\Big|_{x\equiv (n\pm\hat{\mu}/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$

(+) Forward ; (-) Backward

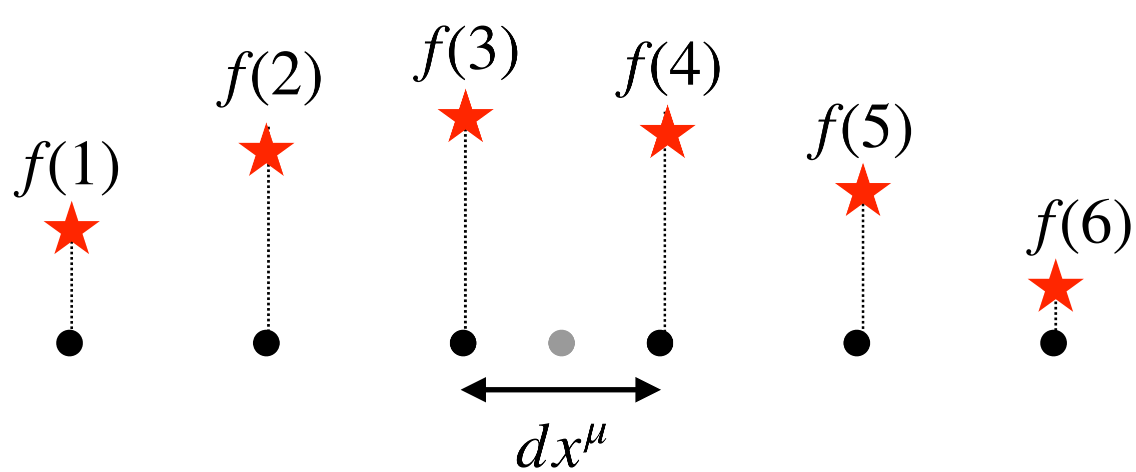
Arbitrary: $$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m) = \sum_{m'} D_i(m') f(l-m')$$

$$\left[\textbf{Neutral: } D_i^0(\mathbf{m}') = \frac{\delta_{\mathbf{m}',-\hat{\imath}} - \delta_{\mathbf{m}',\hat{\imath}}}{2dx}\,, \quad \textbf{Charged: } \begin{cases} D_i^{\pm}(\mathbf{m}') = \dfrac{\pm\delta_{\mathbf{m}',\mp\hat{\imath}/2} \mp \delta_{\mathbf{m}',\pm\hat{\imath}/2}}{dx}\,, & \text{if } \mathbf{l} = \mathbf{n} + \dfrac{\hat{\imath}}{2}\,, \\ D_i^{\pm}(\mathbf{m}') = \dfrac{\pm\delta_{\mathbf{m}',\mp\hat{\imath}} \mp \delta_{\mathbf{m}',\mathbf{0}}}{dx}\,, & \text{if } \mathbf{l} = \mathbf{n}\,, \end{cases} \right]$$

Primer on Lattice Techniques

Lattice Derivatives

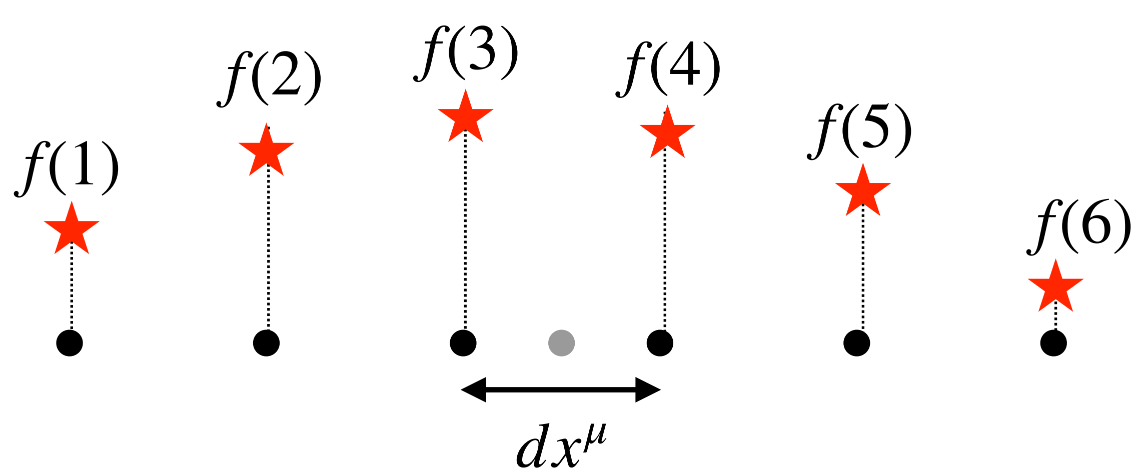
"Goodness" of Lattice representation of continuum fields } controlled by { **Smallness** of dx^μ



Primer on Lattice Techniques

Lattice Derivatives

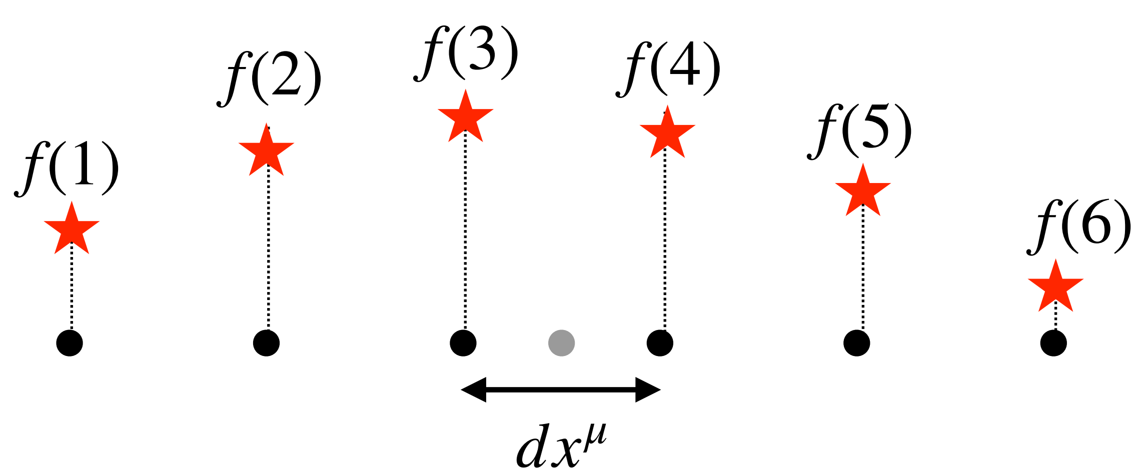
"Goodness" of Lattice representation of continuum fields } controlled by { **Smallness** of dx^{μ}; **Lattice Derivative** Operator ∇_{μ}



Primer on Lattice Techniques

Lattice Derivatives

"Goodness" of Lattice representation of continuum fields } controlled by {
- **Smallness** of dx^μ
- **Lattice Derivative** Operator ∇_μ
- **Location** where operator '*lives*' $n_\mu \pm a_\mu$



Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Derivative: $$\left[\nabla_i f\right](\mathbf{n}) = \sum_{\mathbf{m}'} D_i(\mathbf{m}') f(\mathbf{n} - \mathbf{m}')$$

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative:

$$[\nabla_i f](\tilde{\mathbf{n}}) = \sum_{\mathbf{n}} e^{-\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}} \sum_{\mathbf{m}} D_i(\mathbf{n}-\mathbf{m}) f(\mathbf{m})$$

$$= \sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'} D_i(\mathbf{n}') \sum_{\mathbf{m}} e^{-\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{m}} f(\mathbf{m})$$

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative:

$$[\nabla_i f](\tilde{\mathbf{n}}) = \sum_{\mathbf{n}} e^{-\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}} \sum_{\mathbf{m}} D_i(\mathbf{n}-\mathbf{m}) f(\mathbf{m})$$

$$= \underbrace{\sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'} D_i(\mathbf{n}')}_{} \underbrace{\sum_{\mathbf{m}} e^{-\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{m}} f(\mathbf{m})}_{}$$

$$\equiv \quad -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}}) \qquad\qquad f(\tilde{\mathbf{n}})$$

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative:

$$[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}}) f(\tilde{\mathbf{n}})$$

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative: $[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}})f(\tilde{\mathbf{n}})$

Lattice Momentum:

$$\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'}\vec{D}(\mathbf{n}')$$

weights: define Lattice derivative

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative: $$[\nabla_i f](\tilde{\mathbf{n}}) = -i k_{\mathrm{Lat},i}(\tilde{\mathbf{n}}) f(\tilde{\mathbf{n}})$$

Lattice Momentum: $$\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i \sum_{\mathbf{n}'} e^{\frac{2\pi i}{N} \tilde{\mathbf{n}} \cdot \mathbf{n}'} \vec{D}(\mathbf{n}')$$

Neutral: $$k^0_{\mathrm{Lat},i} = \frac{\sin(2\pi \tilde{n}_i / N)}{dx}$$

Charged: $$\begin{cases} k^+_{\mathrm{Lat},i} = k^-_{\mathrm{Lat},i} = 2\dfrac{\sin(\pi \tilde{n}_i / N)}{dx}, & \text{if } \mathbf{l} = \mathbf{n} \pm \dfrac{\hat{\imath}}{2}, \\ k^\pm_{\mathrm{Lat},i} = \dfrac{\sin(2\pi \tilde{n}_i / N)}{dx} \pm i \dfrac{1 - \cos(2\pi \tilde{n}_i / N)}{dx}, & \text{if } \mathbf{l} = \mathbf{n} \end{cases}$$

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative: $$[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}})f(\tilde{\mathbf{n}})$$

↓

Lattice Momentum: $$\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'}\vec{D}(\mathbf{n}')$$

Linear: $$k_{\mathrm{Lin},i} = \tilde{n}_i\frac{2\pi}{Ndx}\,,$$

Charged: $$k^{+}_{\mathrm{Lat},i} = k^{-}_{\mathrm{Lat},i} = 2\frac{\sin(\pi\tilde{n}_i/N)}{dx}\,,$$

(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)

Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

Fourier transform of a Derivative: $[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}})f(\tilde{\mathbf{n}})$

Lattice Momentum:

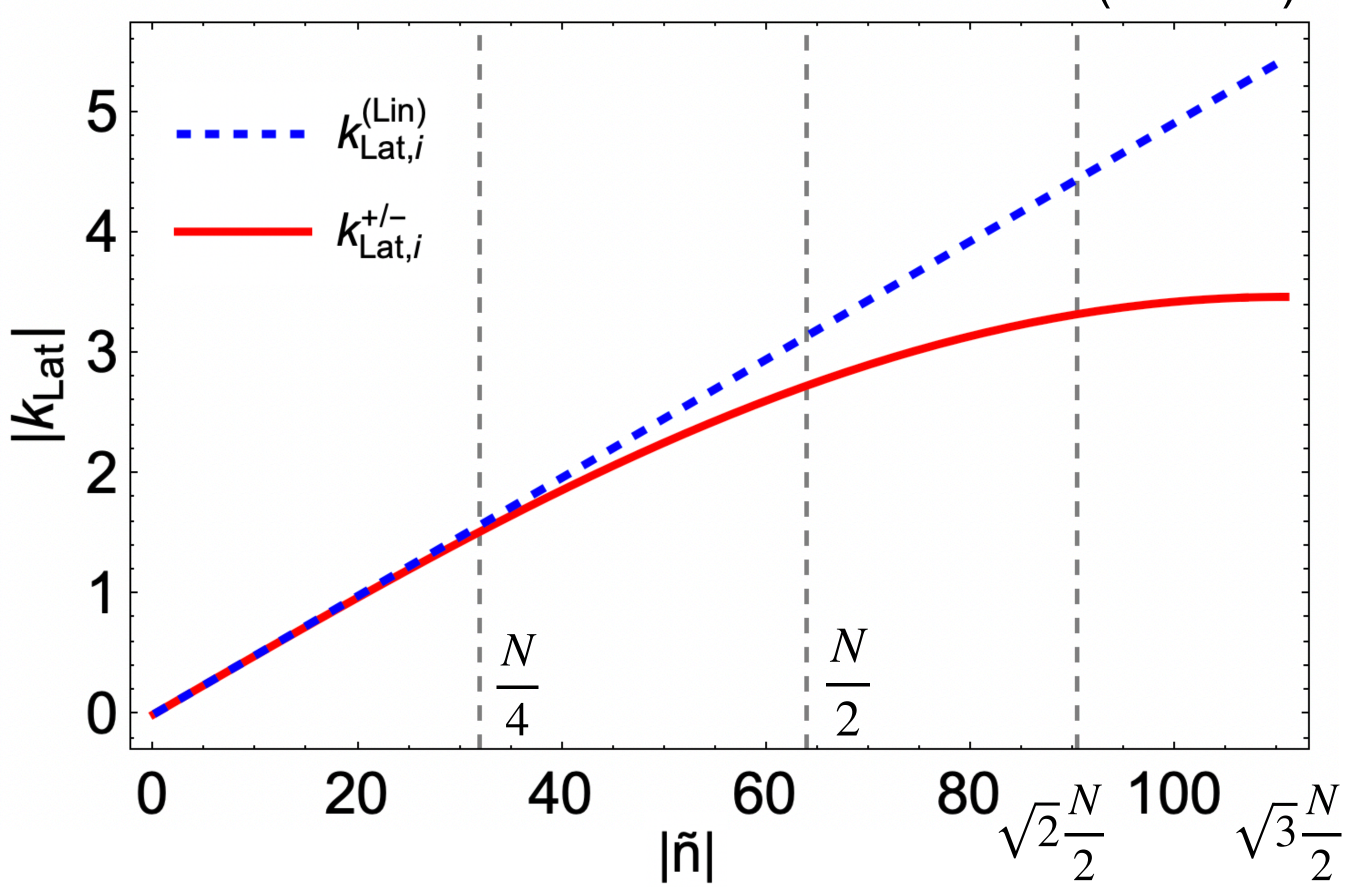
$$\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'}\vec{D}(\mathbf{n}')$$

Linear: $k_{\mathrm{Lin},i} = \tilde{n}_i \dfrac{2\pi}{Ndx}\,,$

Charged: $k^{+}_{\mathrm{Lat},i} = k^{-}_{\mathrm{Lat},i} = 2\dfrac{\sin(\pi\tilde{n}_i/N)}{dx}\,,$

(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)

(N =128)



Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

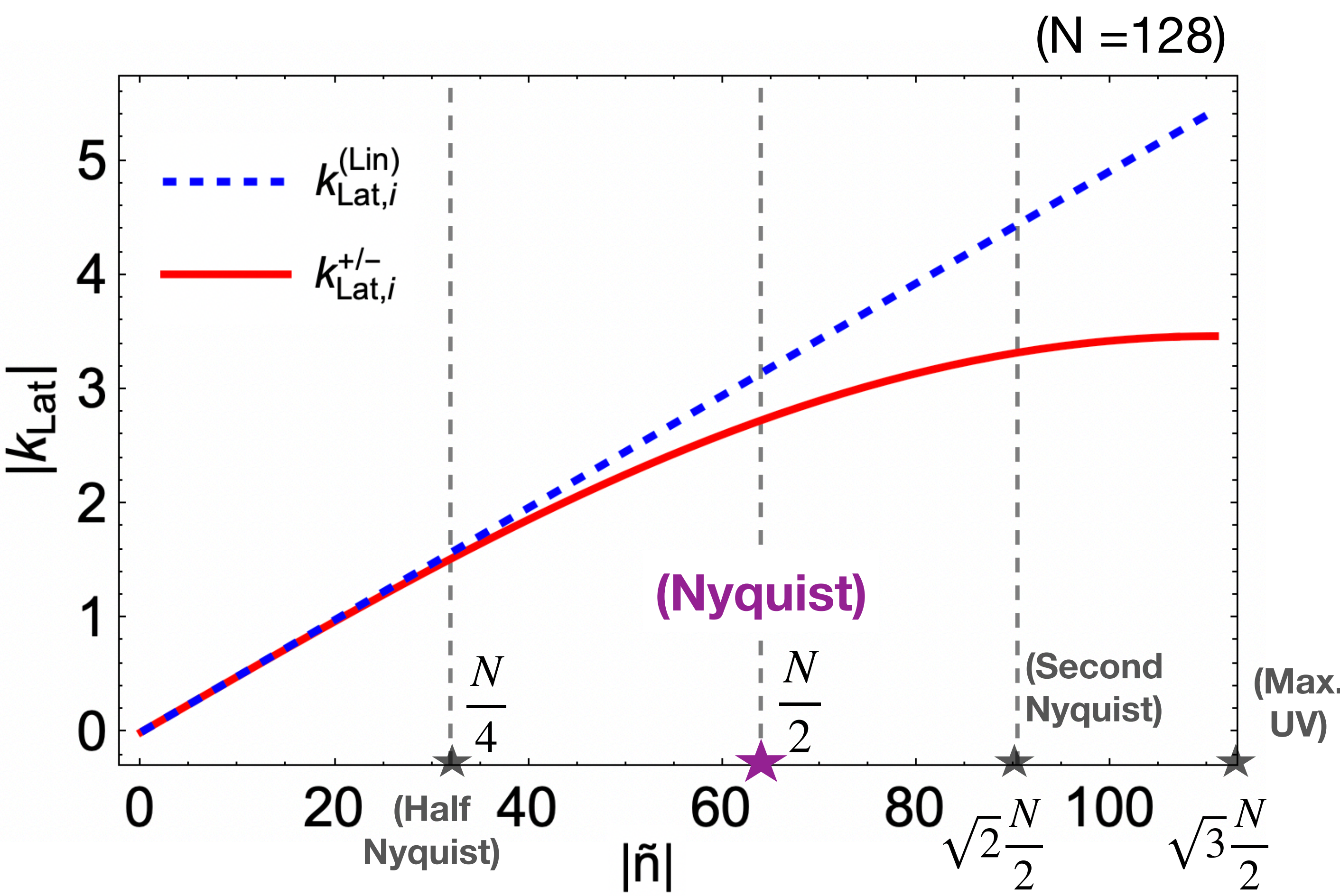
Fourier transform of a Derivative: $[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\text{Lat},i}(\tilde{\mathbf{n}})f(\tilde{\mathbf{n}})$

Lattice Momentum:
$$\mathbf{k}_{\text{Lat}}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'}\vec{D}(\mathbf{n}')$$

Lineal: $k_{\text{Lin},i} = \tilde{n}_i\dfrac{2\pi}{Ndx}\,,$

Charged: $k^{+}_{\text{Lat},i} = k^{-}_{\text{Lat},i} = 2\dfrac{\sin(\pi\tilde{n}_i/N)}{dx}\,,$

(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)



Primer on Lattice Techniques

Lattice Derivatives → Lattice Momentum

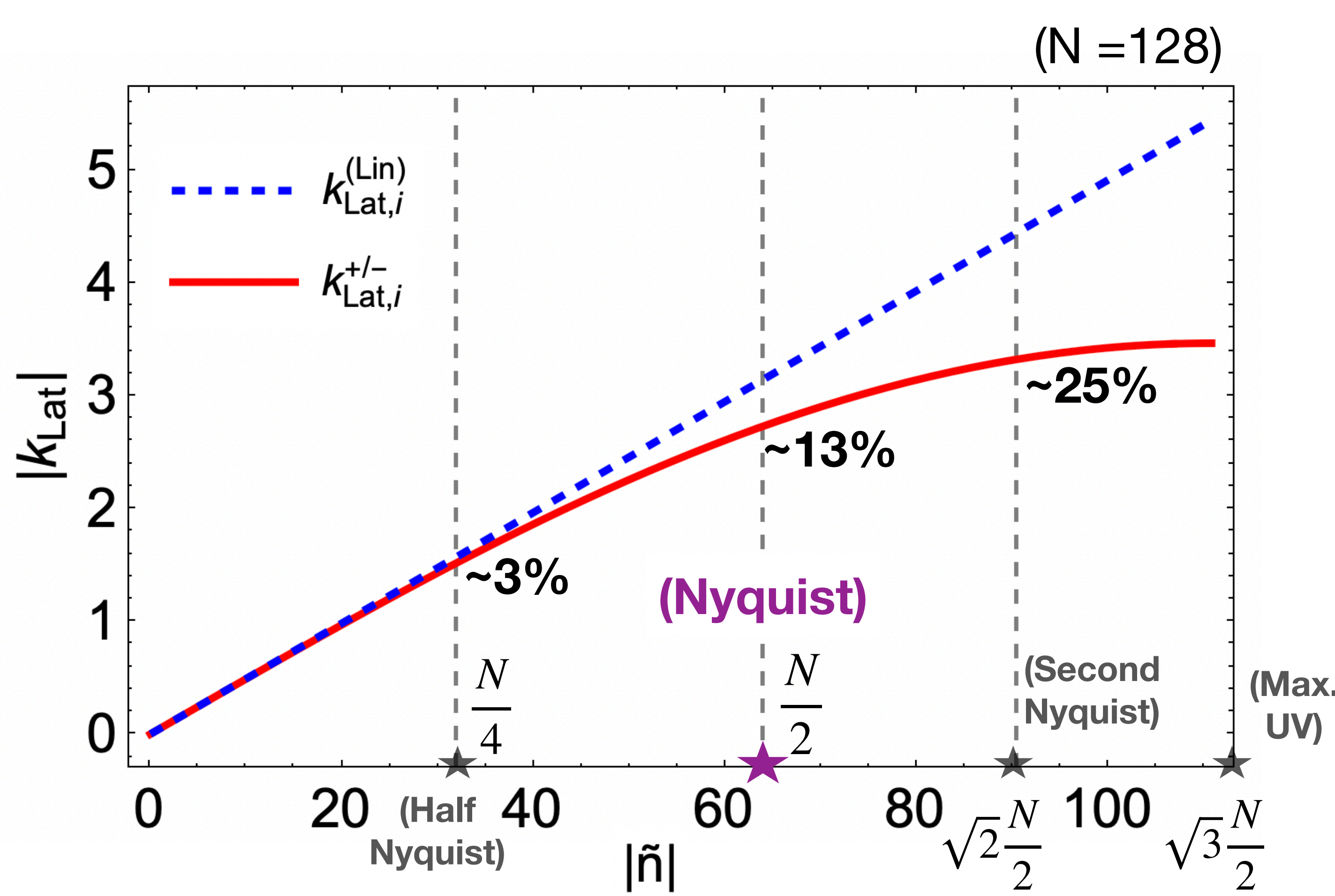
Fourier transform of a Derivative: $[\nabla_i f](\tilde{\mathbf{n}}) = -ik_{\text{Lat},i}(\tilde{\mathbf{n}}) f(\tilde{\mathbf{n}})$

Lattice Momentum: $$\mathbf{k}_{\text{Lat}}(\tilde{\mathbf{n}}) = i \sum_{\mathbf{n}'} e^{\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'} \vec{D}(\mathbf{n}')$$

Lineal: $k_{\text{Lin},i} = \tilde{n}_i \frac{2\pi}{Ndx}$,

Charged: $k^{+}_{\text{Lat},i} = k^{-}_{\text{Lat},i} = 2\frac{\sin(\pi\tilde{n}_i/N)}{dx}$,

(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)



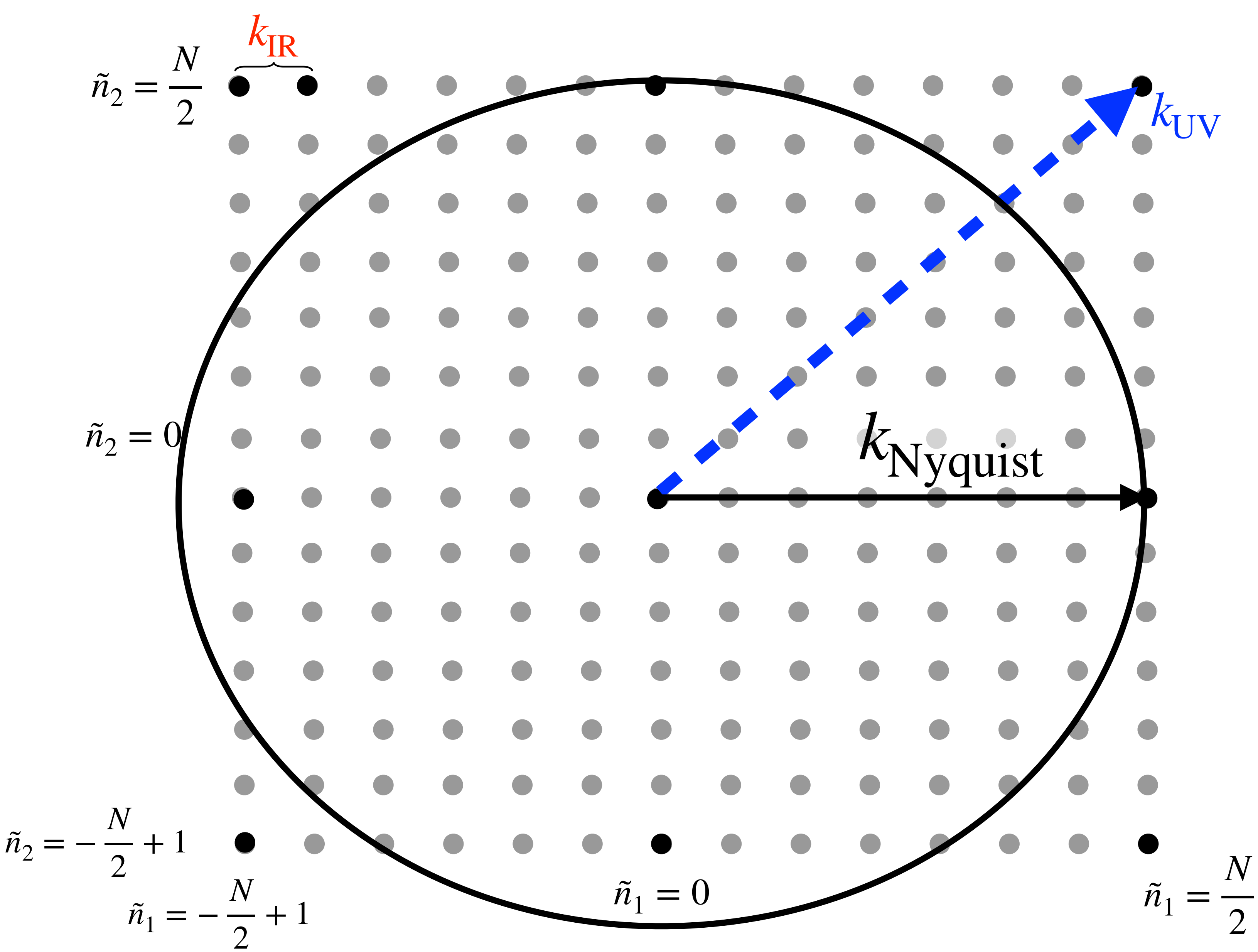
Primer on Lattice Techniques

Definition of Fourier Lattice

Momentum Lattice Periodic

$$\tilde{n}_i = -\frac{N}{2}+1, \dots, -1, 0, 1, \dots, \frac{N}{2}$$

$$f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \quad (i = 1,2,\dots,d)$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$

Primer on Lattice Techniques

Definition of Power Spectrum

$$\mathtt{f}(\mathbf{x},t) = F(t) + \delta F(\mathbf{x},t) \quad \longrightarrow \quad \mathtt{f}_{\mathbf{k}}(t)$$

Function with random amplitudes (↑ $\delta F(\mathbf{x},t)$)

Fourier Transform (↑ $\mathtt{f}_{\mathbf{k}}(t)$)

Primer on Lattice Techniques

Definition of Power Spectrum

$$\mathtt{f}(\mathbf{x},t) = F(t) + \delta F(\mathbf{x},t) \longrightarrow \mathtt{f}_{\mathbf{k}}(t)$$

Function with random amplitudes

Fourier Transform

$$\langle \mathtt{f}^2(\mathbf{x},t)\rangle = \int d\log k \;\; \Delta_{\mathtt{f}}(k,t) \;\; ,$$

Ensemble Average

Primer on Lattice Techniques

Definition of Power Spectrum

$$\mathtt{f}(\mathbf{x},t) = F(t) + \delta F(\mathbf{x},t) \longrightarrow \mathtt{f}_{\mathbf{k}}(t)$$

Function with random amplitudes

Fourier Transform

$$\langle \mathtt{f}^2(\mathbf{x},t)\rangle = \int d\log k \; \Delta_{\mathtt{f}}(k,t) \;,$$

Ensemble Average

$$\Delta_{\mathtt{f}}(k,t) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k,t)$$

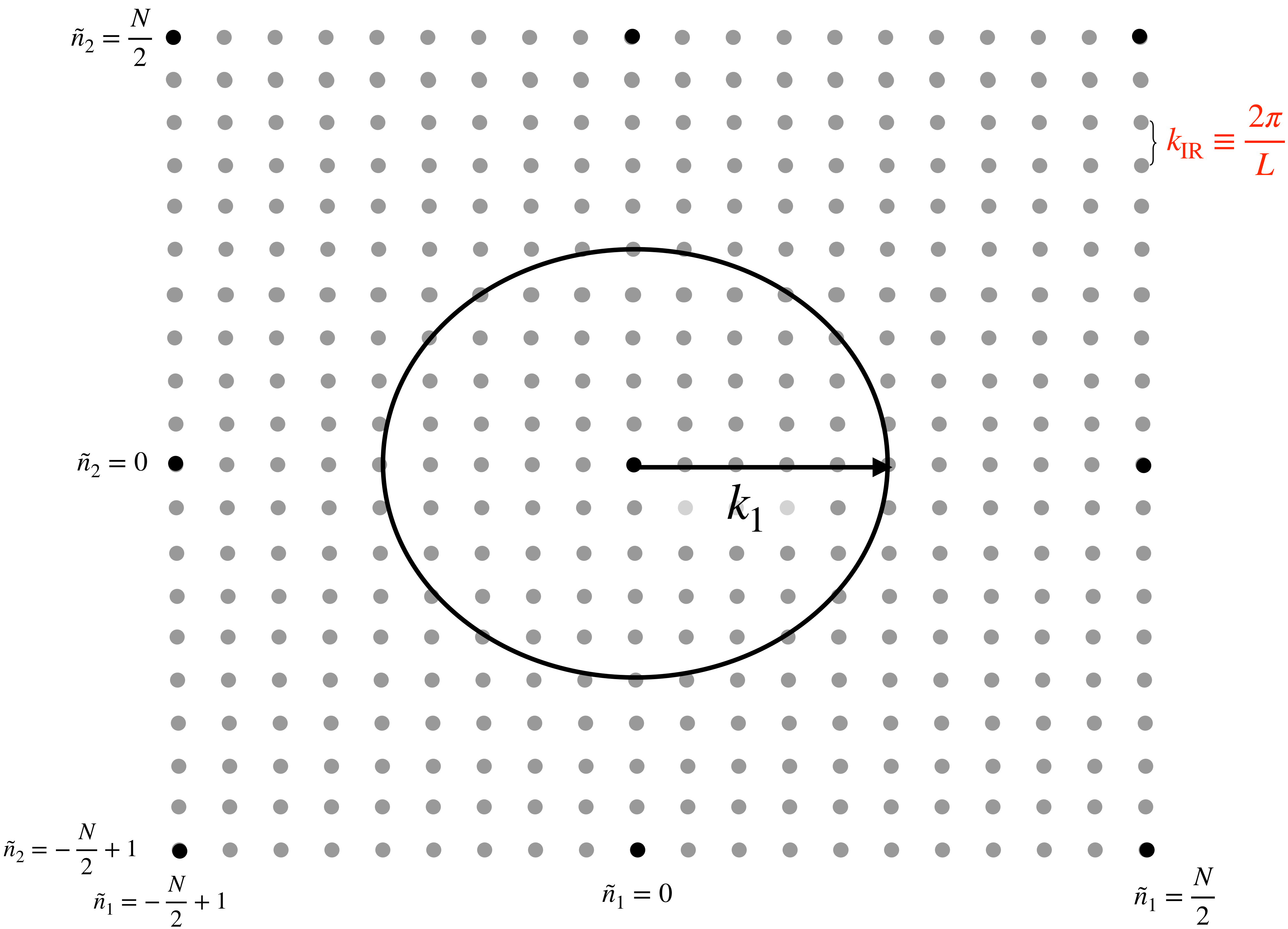
Power Spectrum

$$\langle \mathtt{f}_{\mathbf{k}}(t)\mathtt{f}^*_{\mathbf{k}'}(t)\rangle = (2\pi)^3\mathscr{P}_{\mathtt{f}}(k,t)\delta(\mathbf{k}-\mathbf{k}')$$

2-point Funct.

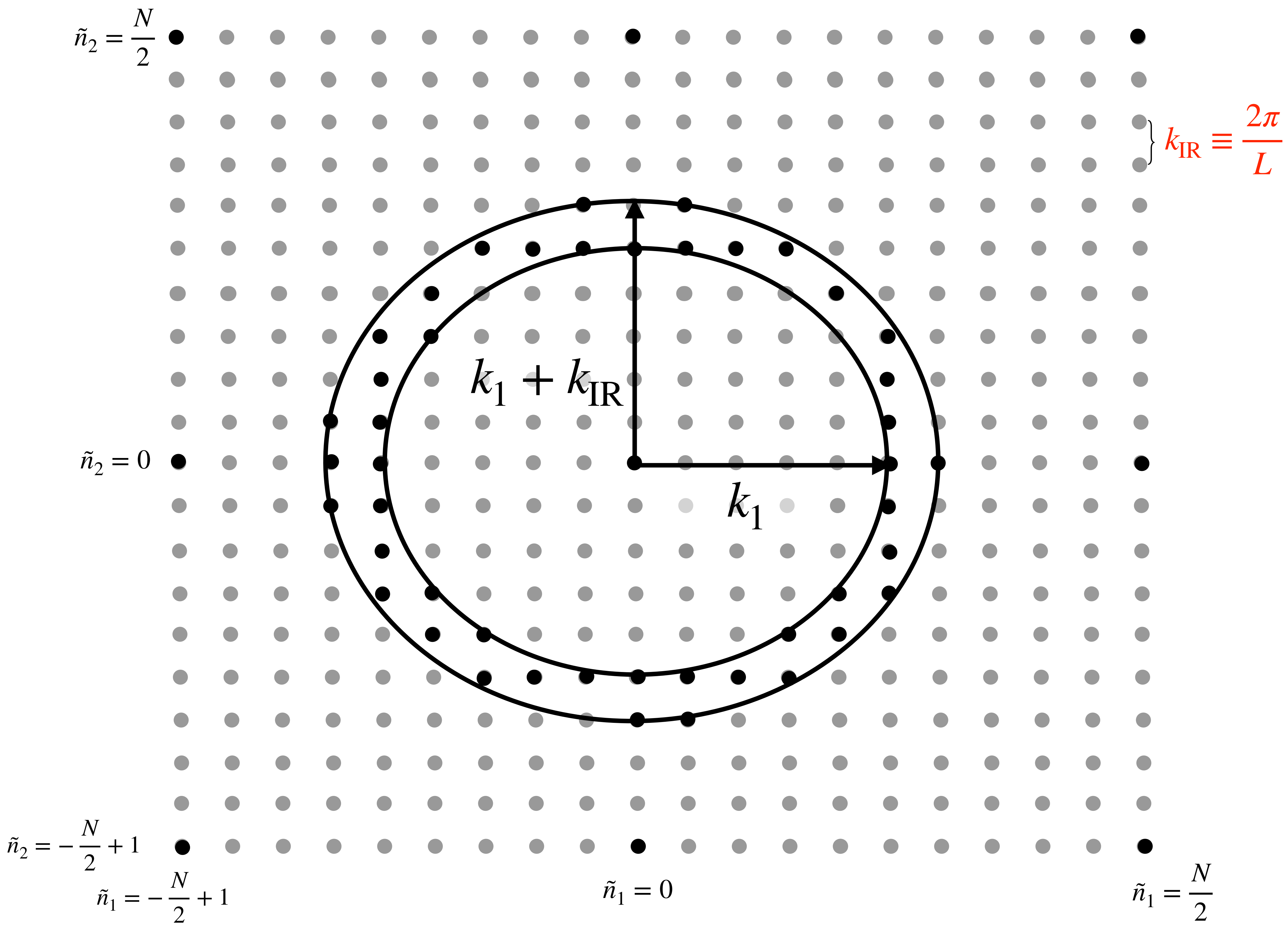
Primer on Lattice Techniques

Definition of Power Spectrum



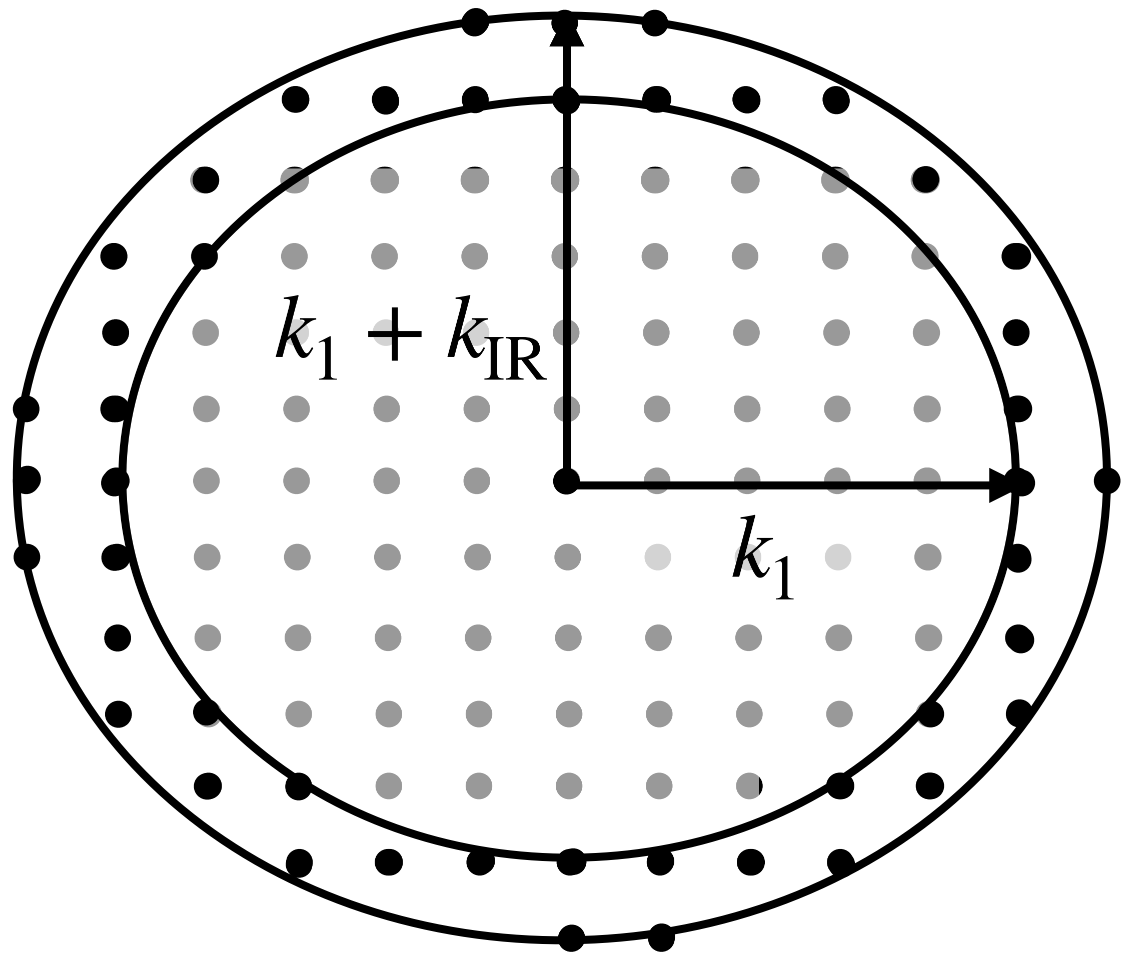
Primer on Lattice Techniques

Definition of Power Spectrum



Primer on Lattice Techniques

Definition of Power Spectrum



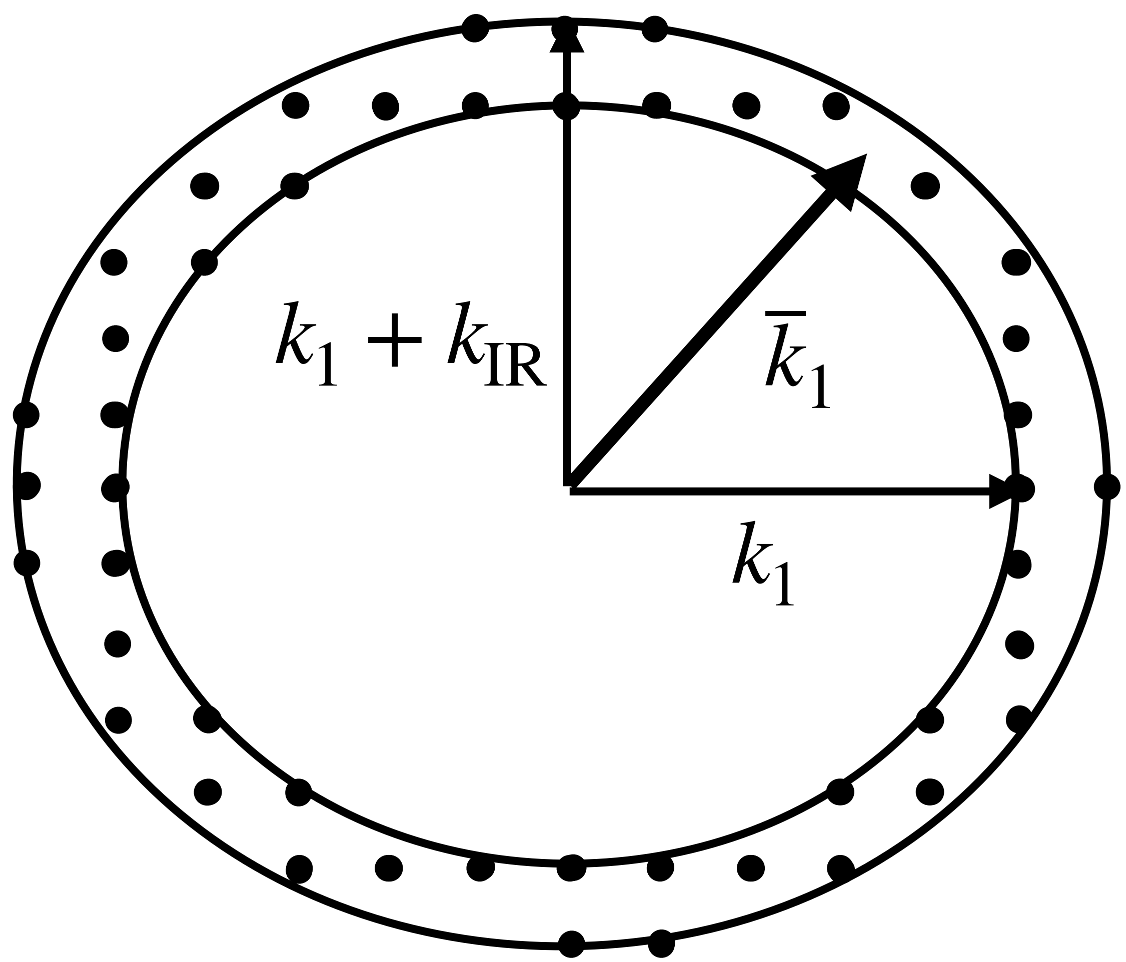
Primer on Lattice Techniques

Definition of Power Spectrum

(multiplicity)

Bin 1: $k(\tilde{\mathbf{n}}) \in [k_1, k_1 + k_{\rm IR}]$ ⟶ $\{\mathtt{f}_{\mathbf{k}}\}\ ;\quad \#_1 \simeq \Omega_d n_1^{d-1}\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

(d-dimensional solid angle in d spatial dims)



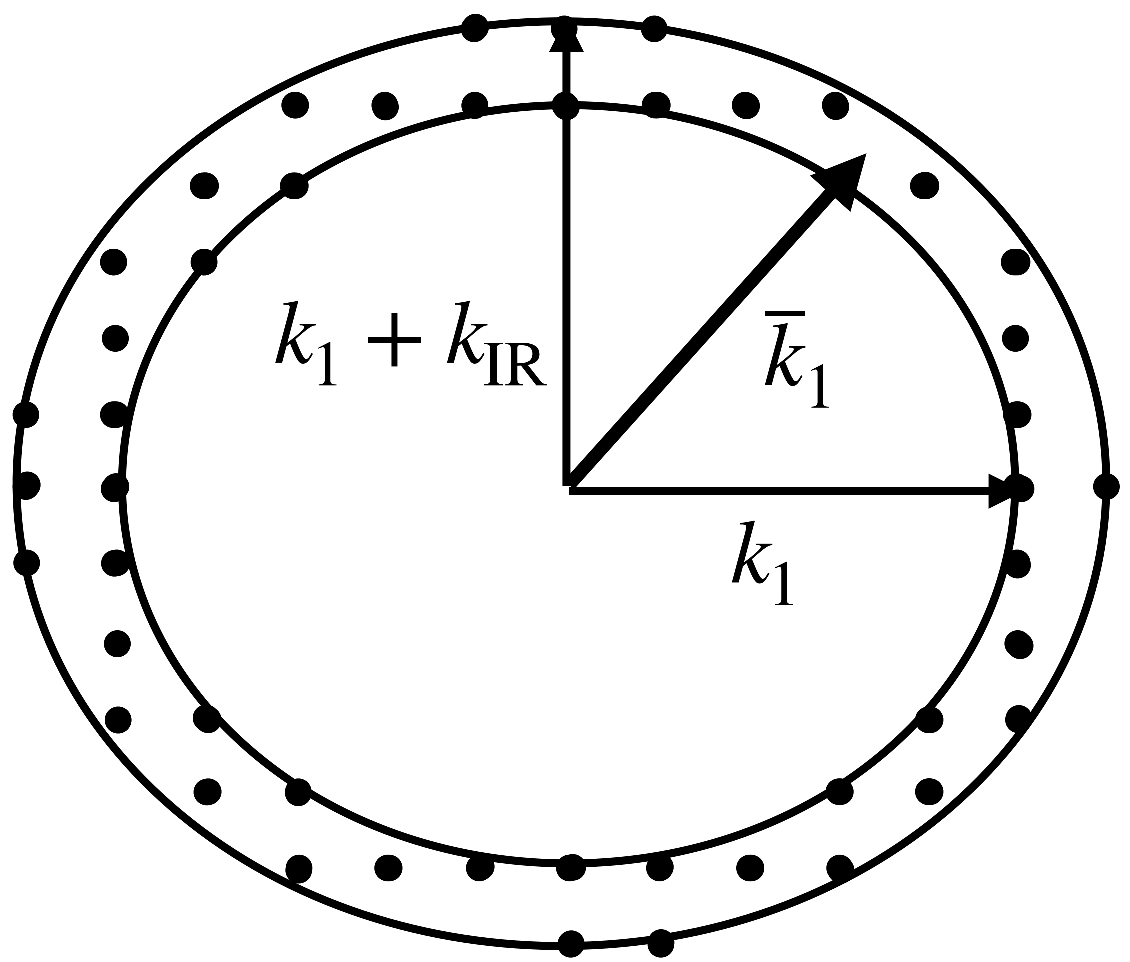
Primer on Lattice Techniques

Definition of Power Spectrum

(multiplicity)

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1} \longrightarrow \{\texttt{f}_{\mathbf{k}}\}\ ;\quad \#_1 \simeq 4\pi n_1^2\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

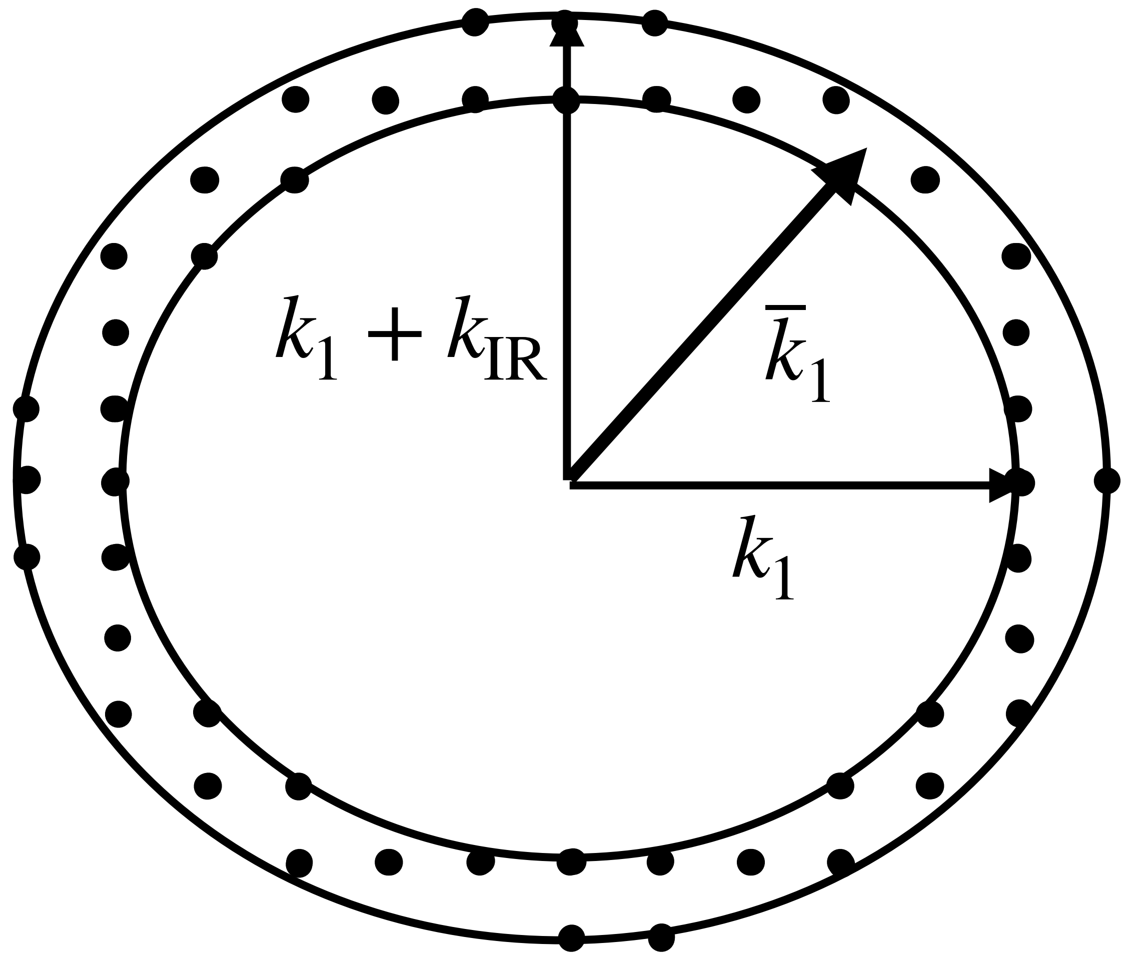


Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1} \longrightarrow \{\mathtt{f}_{\mathbf{k}}\}\ ;\quad \#_1 \simeq 4\pi n_1^2\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

(multiplicity) (d = 3)



$$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle |\mathtt{f}_{\mathbf{k}}|^2 \rangle_{R_1}$$

(angular average)

≡

(ensemble average)

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}\ ;\quad \#_1 \simeq 4\pi n_1^2\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}}\mathtt{f}^*_{\mathbf{k}}\rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}}\mathtt{f}^*_{\mathbf{k}'}\rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

---

*

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}\ ;\quad \#_1 \simeq 4\pi n_1^2\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k\ \Delta_{\mathtt{f}}(k)\ ,\quad \Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)\ ,\quad \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}\ ;\quad \underset{(\mathrm{d}=3)}{\#_1 \simeq 4\pi n_1^2}\ ;\quad \bar{k}_1 \simeq n_1 k_{\rm IR}$

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathcal{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k\ \Delta_{\mathtt{f}}(k)\ ,\quad \Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathcal{P}_{\mathtt{f}}(k)\ ,\quad \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathcal{P}_{\mathtt{f}}(k)\delta(\mathbf{k}-\mathbf{k}')$

Lattice: $\langle f^2 \rangle_V = \frac{dx^3}{L^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^6}\sum_{\tilde{\mathbf{n}}} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2$

↑ **Lattice FT**

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$; $\bar{k}_1 \simeq n_1 k_{\rm IR}$
(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

$\longrightarrow$ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \, \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Lattice:

$$\langle f^2 \rangle_V = \frac{dx^3}{L^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^6}\sum_{\tilde{\mathbf{n}}} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2$$

$$= \frac{1}{N^6}\sum_{|\tilde{\mathbf{n}}|} \underbrace{\sum_{\tilde{\mathbf{n}}' \in R(\tilde{\mathbf{n}})} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2}$$

↑ **Radial-Angular Decomposition**

Angular Summation

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\underset{(\mathrm{d}=3)}{\#_1 \simeq 4\pi n_1^2}$; $\bar{k}_1 \simeq n_1 k_{\rm IR}$

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \; \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \dfrac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Lattice:

$$\langle f^2 \rangle_V = \frac{dx^3}{L^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^6}\sum_{\tilde{\mathbf{n}}} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2$$

$$= \frac{1}{N^6}\sum_{|\tilde{\mathbf{n}}|} \underbrace{\sum_{\tilde{\mathbf{n}}' \in R(\tilde{\mathbf{n}})} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2}_{} \simeq \frac{4\pi}{N^6}\sum_{|\tilde{\mathbf{n}}|} |\tilde{\mathbf{n}}|^2 \left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

Radial-Angular Decomposition

Angular Summation

Multiplicity $\simeq 4\pi|\tilde{\mathbf{n}}|^2$

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$; $\bar{k}_1 \simeq n_1 k_{IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

→ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \, \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \dfrac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Lattice: $\langle f^2 \rangle_V \simeq \dfrac{4\pi}{N^6} \sum_{|\tilde{\mathbf{n}}|} |\tilde{\mathbf{n}}|^2 \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$

Summation over bins

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$ (d = 3) ; $\bar{k}_1 \simeq n_1 k_{\rm IR}$

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

→ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k\ \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \dfrac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Lattice: $\langle f^2 \rangle_V \simeq \dfrac{4\pi}{N^6} \sum_{|\tilde{\mathbf{n}}|} |\tilde{\mathbf{n}}|^2 \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} = \sum_{|\tilde{\mathbf{n}}|} \Delta \log k(\tilde{\mathbf{n}}) \dfrac{k^3(\tilde{\mathbf{n}})}{2\pi^2} \dfrac{L^3}{N^6} \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$

Summation over bins

$\Delta \log k(\tilde{\mathbf{n}}) \equiv \dfrac{k_{\rm IR}}{k(\tilde{\mathbf{n}})}$; $k(\tilde{\mathbf{n}}) \equiv k_{\rm IR} |\tilde{\mathbf{n}}|$

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$; $\bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

$\longrightarrow$ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d \log k \, \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2} \mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k) \delta(\mathbf{k} - \mathbf{k}')$

Lattice: $\langle f^2 \rangle_V \simeq \frac{4\pi}{N^6} \sum_{|\tilde{\mathbf{n}}|} |\tilde{\mathbf{n}}|^2 \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} = \sum_{|\tilde{\mathbf{n}}|} \Delta \log k(\tilde{\mathbf{n}}) \underbrace{\frac{k^3(\tilde{\mathbf{n}})}{2\pi^2} \frac{L^3}{N^6} \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}}_{\text{Lattice Power Spectrum}}$

Summation over bins

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$; $\bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \; \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2} \mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k) \delta(\mathbf{k} - \mathbf{k}')$

Lattice: $\langle f^2 \rangle_V \simeq \sum_{|\tilde{\mathbf{n}}|} \Delta \log k(\tilde{\mathbf{n}}) \; \Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|)$, (Summation over bins)

Primer on Lattice Techniques

Definition of Power Spectrum

Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

(multiplicity)

$\{\mathtt{f}_{\mathbf{k}}\}$; $\#_1 \simeq 4\pi n_1^2$; $\bar{k}_1 \simeq n_1 k_{\rm IR}$

(d = 3)

$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}} \rangle_{R_1}$

(angular average)

⟶ **Related to** $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle \propto \mathscr{P}_{\mathtt{f}}(k)$ **?**

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \, \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k} - \mathbf{k}')$

Lattice: $\langle f^2 \rangle_V \simeq \sum_{|\tilde{\mathbf{n}}|} \Delta \log k(\tilde{\mathbf{n}}) \, \Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|)$, (Summation over bins)

Lattice Power Spectrum: $\Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2} \underbrace{\frac{L^3}{N^6} \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}}$

Lattice representation of continuum $\langle \mathtt{f}_{\mathbf{k}} \mathtt{f}^*_{\mathbf{k}'} \rangle$

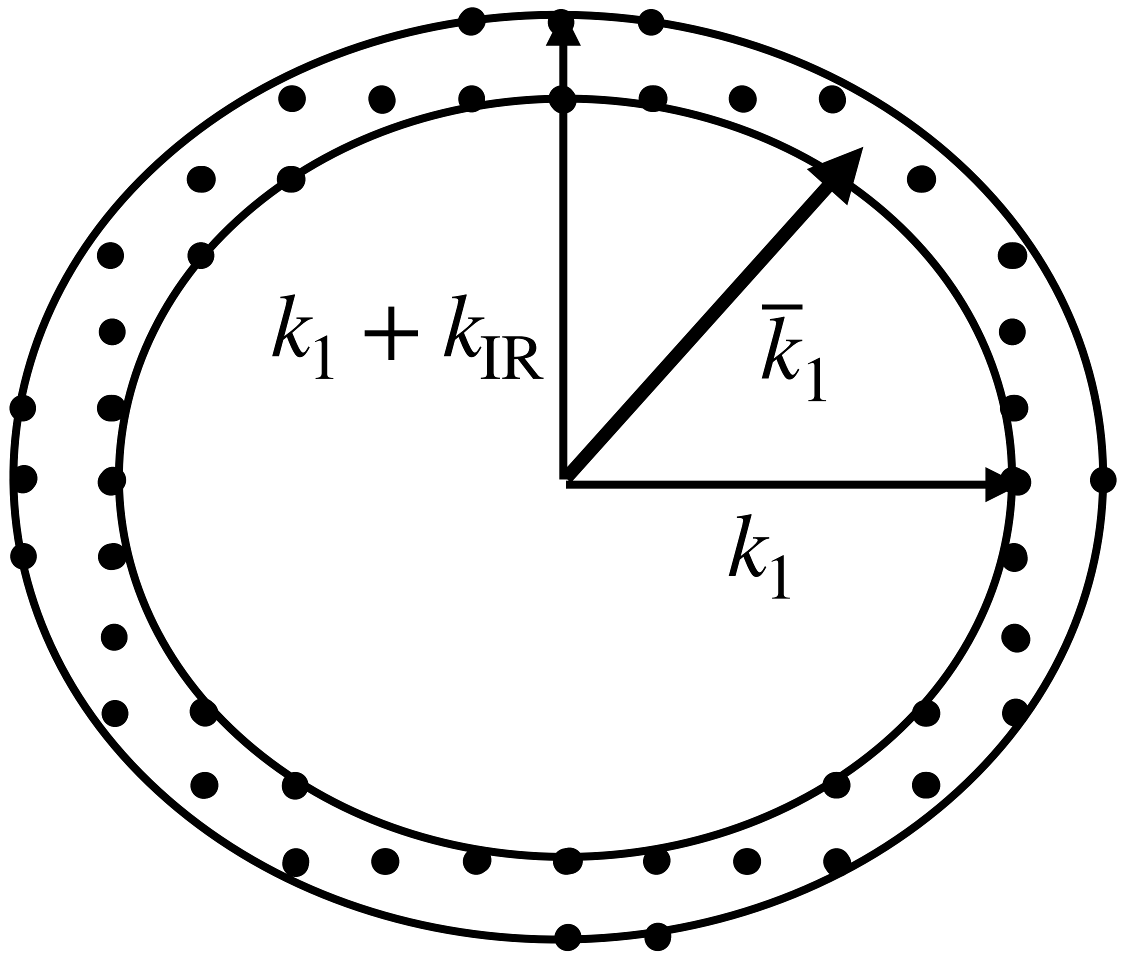
Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2\right\rangle_{R(\tilde{\mathbf{n}})}$$

Based on: $\langle(\cdots)\rangle_{R(\tilde{\mathbf{n}})} \equiv \frac{1}{4\pi|\tilde{\mathbf{n}}|^2}\sum_{\tilde{\mathbf{n}}'\in R(\tilde{\mathbf{n}})}(\cdots)$

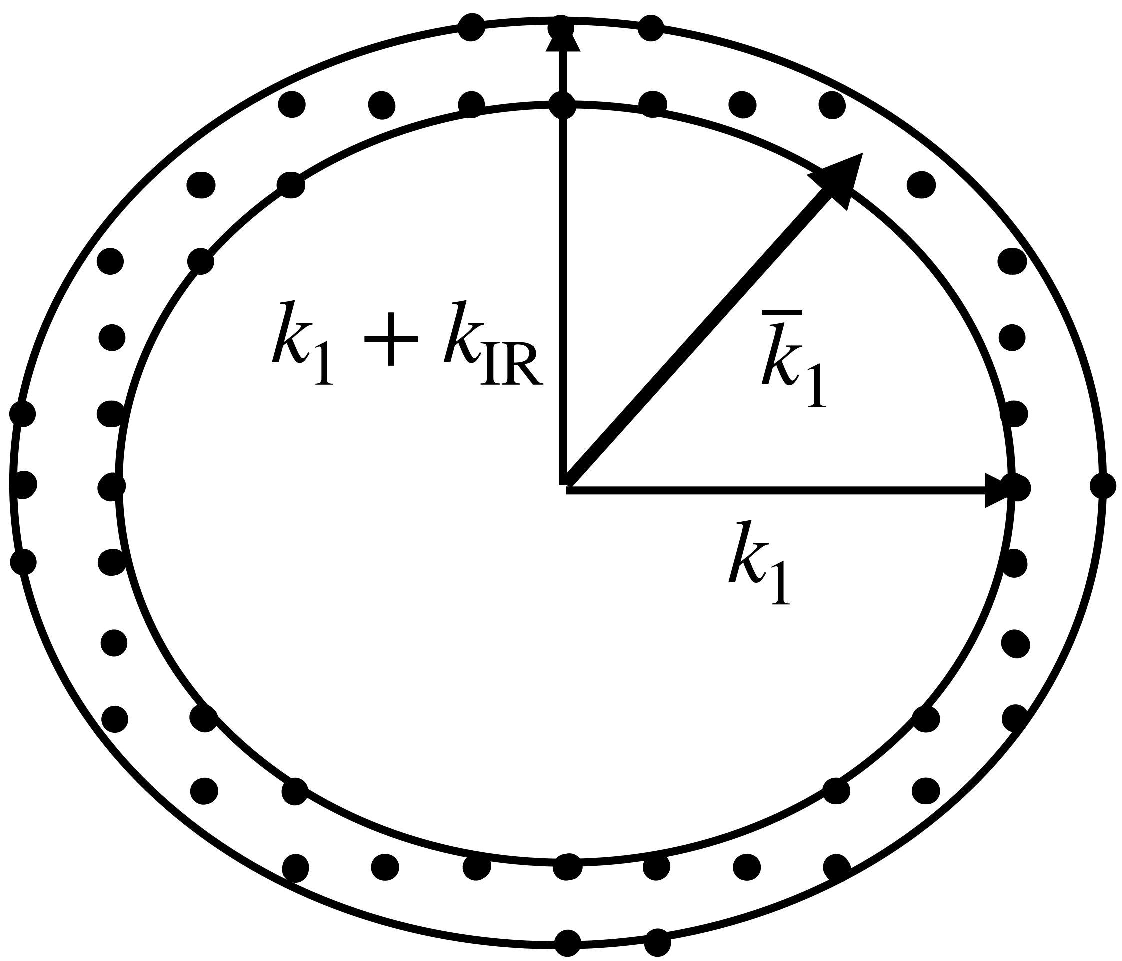
Multiplicity $\simeq 4\pi|\tilde{\mathbf{n}}|^2$



Primer on Lattice Techniques

Definition of Power Spectrum

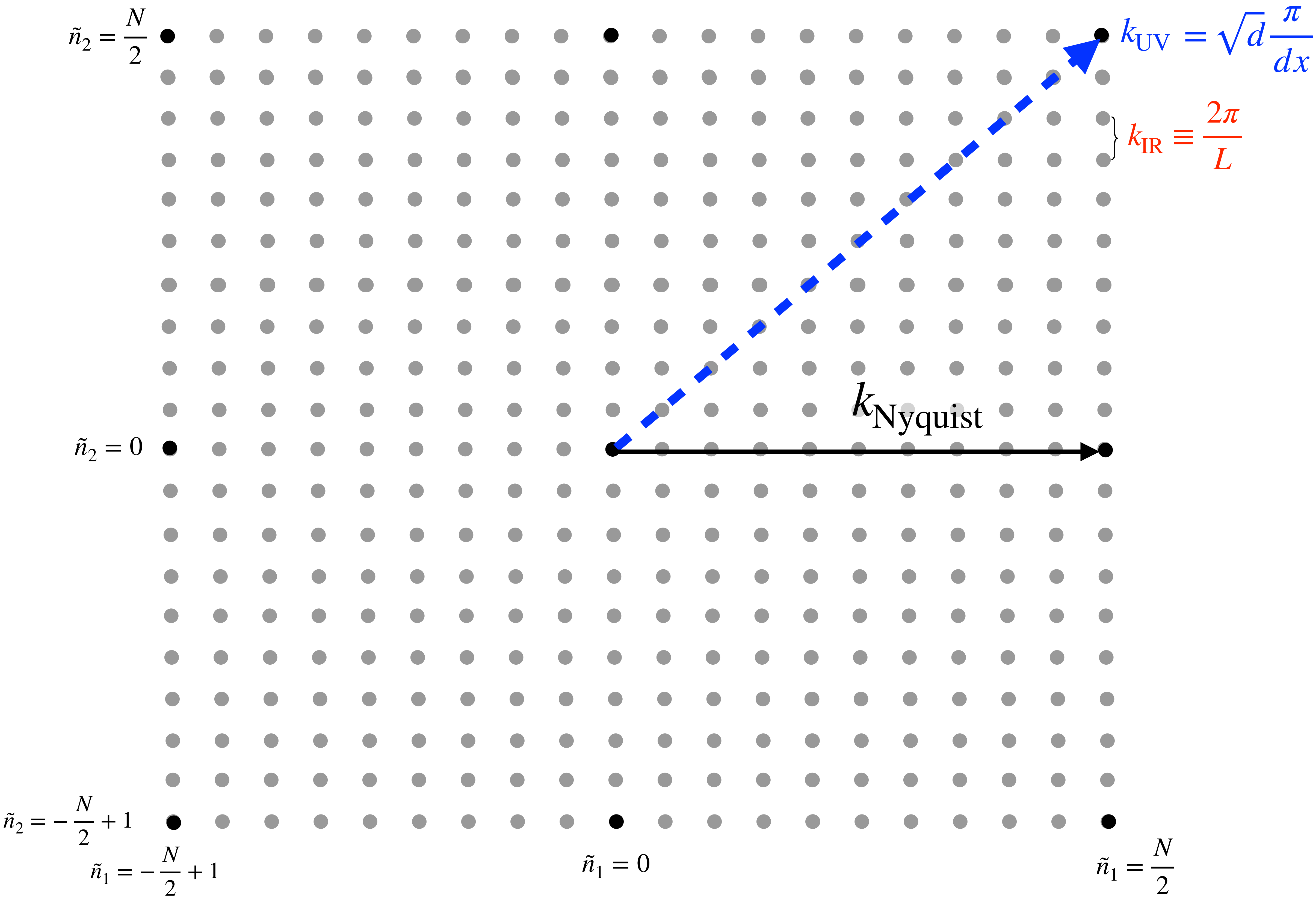
$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} \qquad \left(\begin{array}{c} \textbf{Multiplicity} \\ \simeq 4\pi|\tilde{\mathbf{n}}|^2 \end{array} \right)$$



Primer on Lattice Techniques

Definition of Power Spectrum

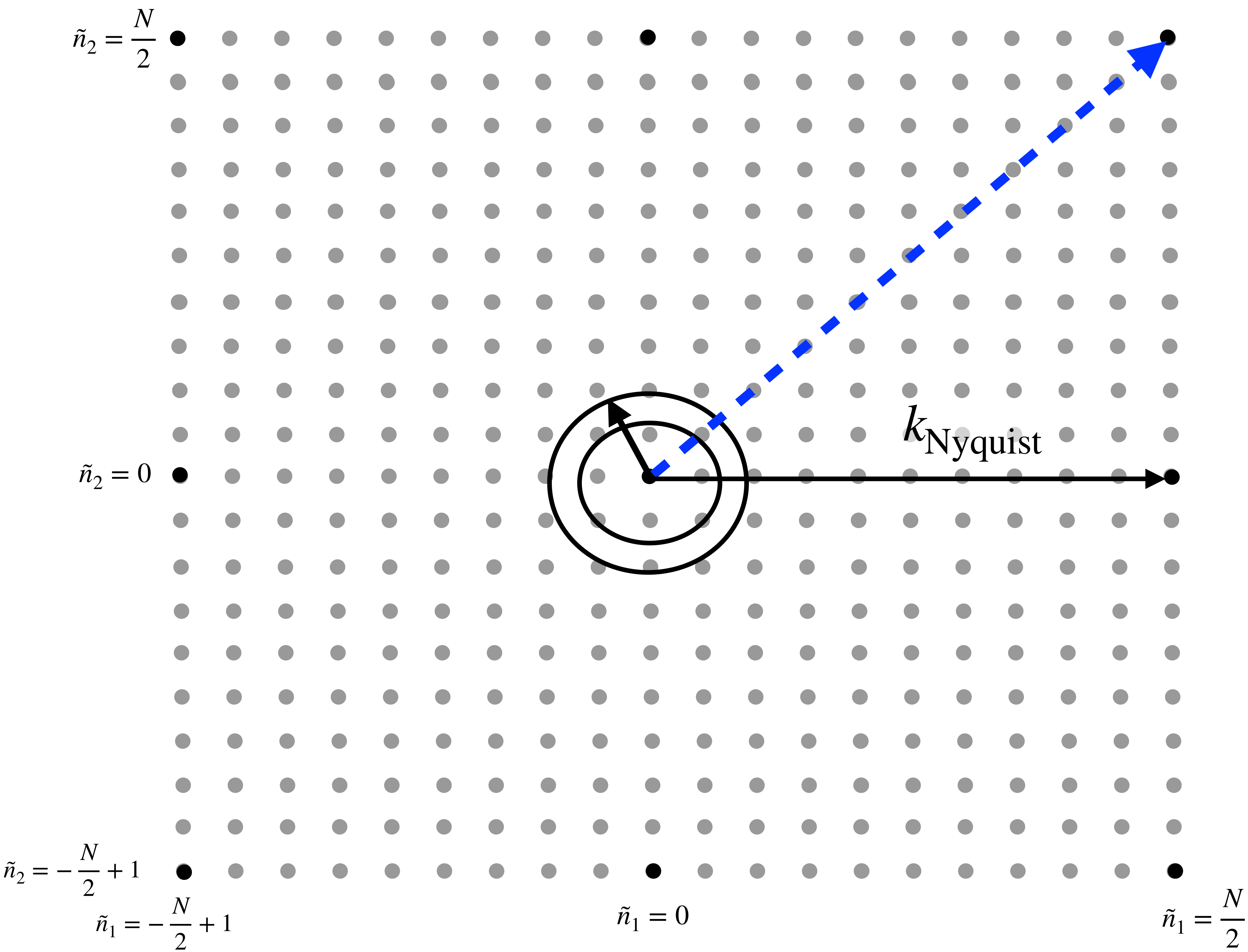
$\left(\begin{array}{c} \textbf{Multiplicity} \\ \simeq 4\pi |\tilde{\mathbf{n}}|^2 \end{array} \right)$



Primer on Lattice Techniques

Definition of Power Spectrum

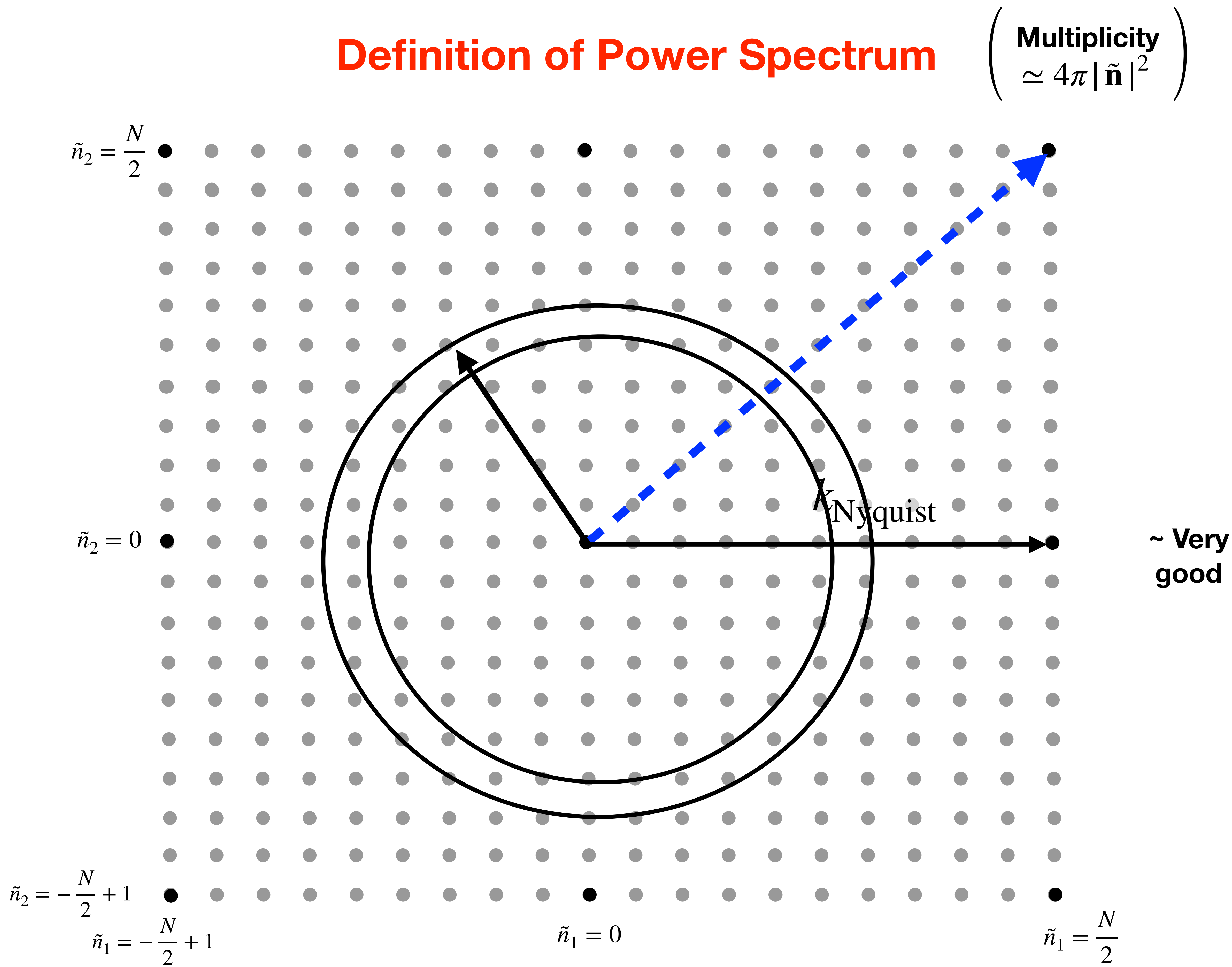
$\left(\begin{array}{c} \textbf{Multiplicity} \\ \simeq 4\pi |\tilde{\mathbf{n}}|^2 \end{array} \right)$



~ works
Ok-ish

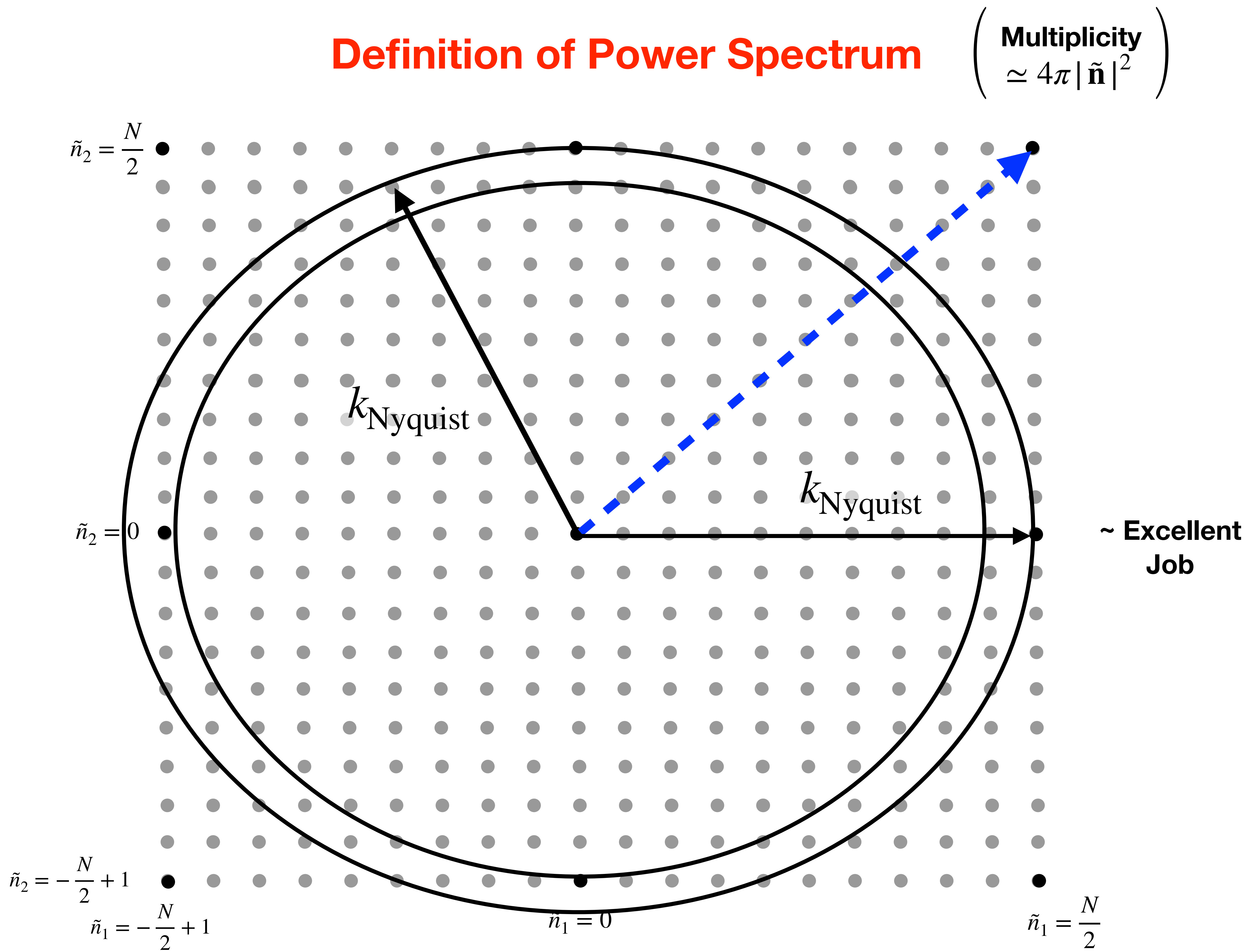
Primer on Lattice Techniques

Definition of Power Spectrum



Primer on Lattice Techniques

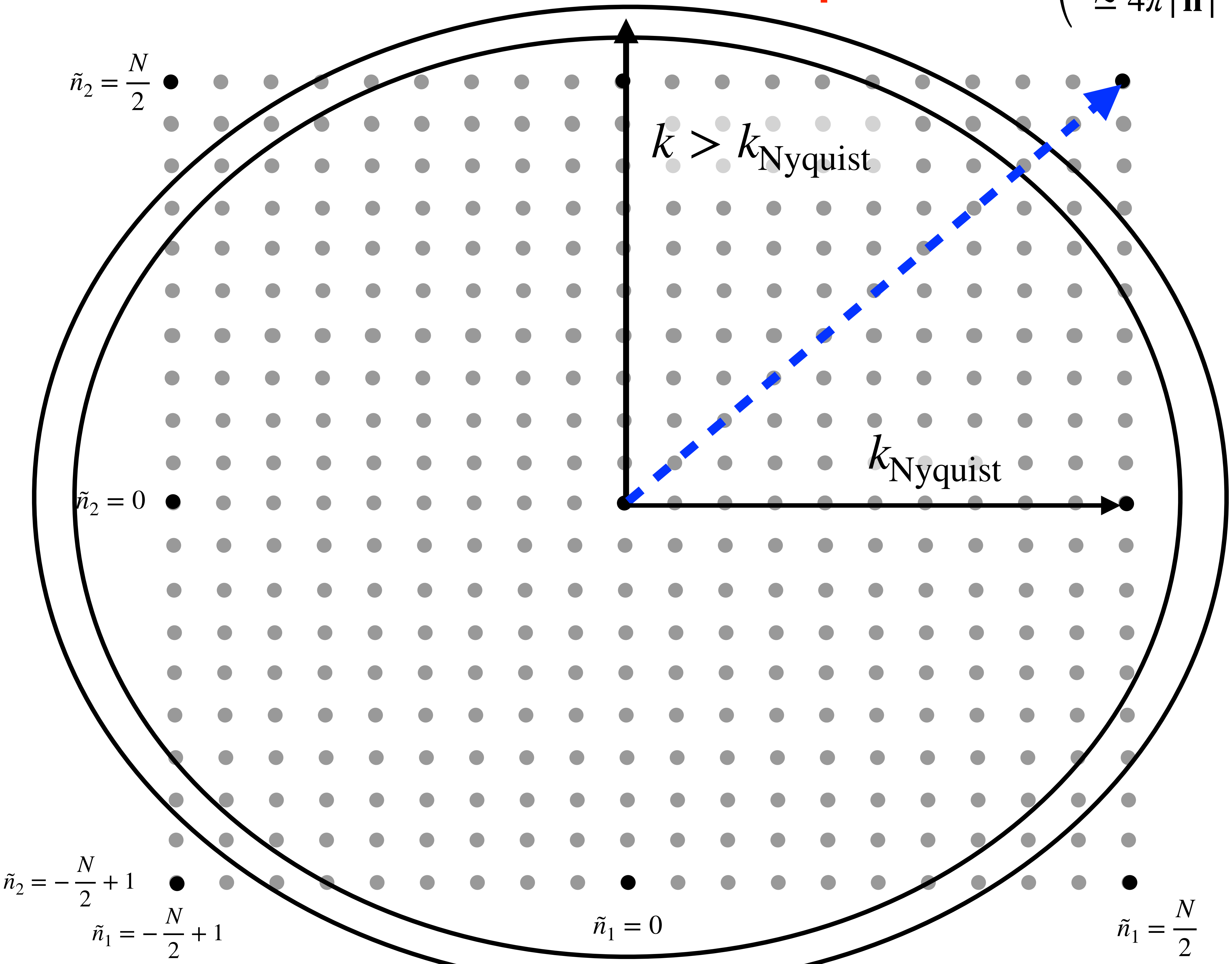
Definition of Power Spectrum



Primer on Lattice Techniques

Definition of Power Spectrum

$\left(\begin{array}{c} \textbf{Multiplicity} \\ \simeq 4\pi |\tilde{\mathbf{n}}|^2 \end{array} \right)$



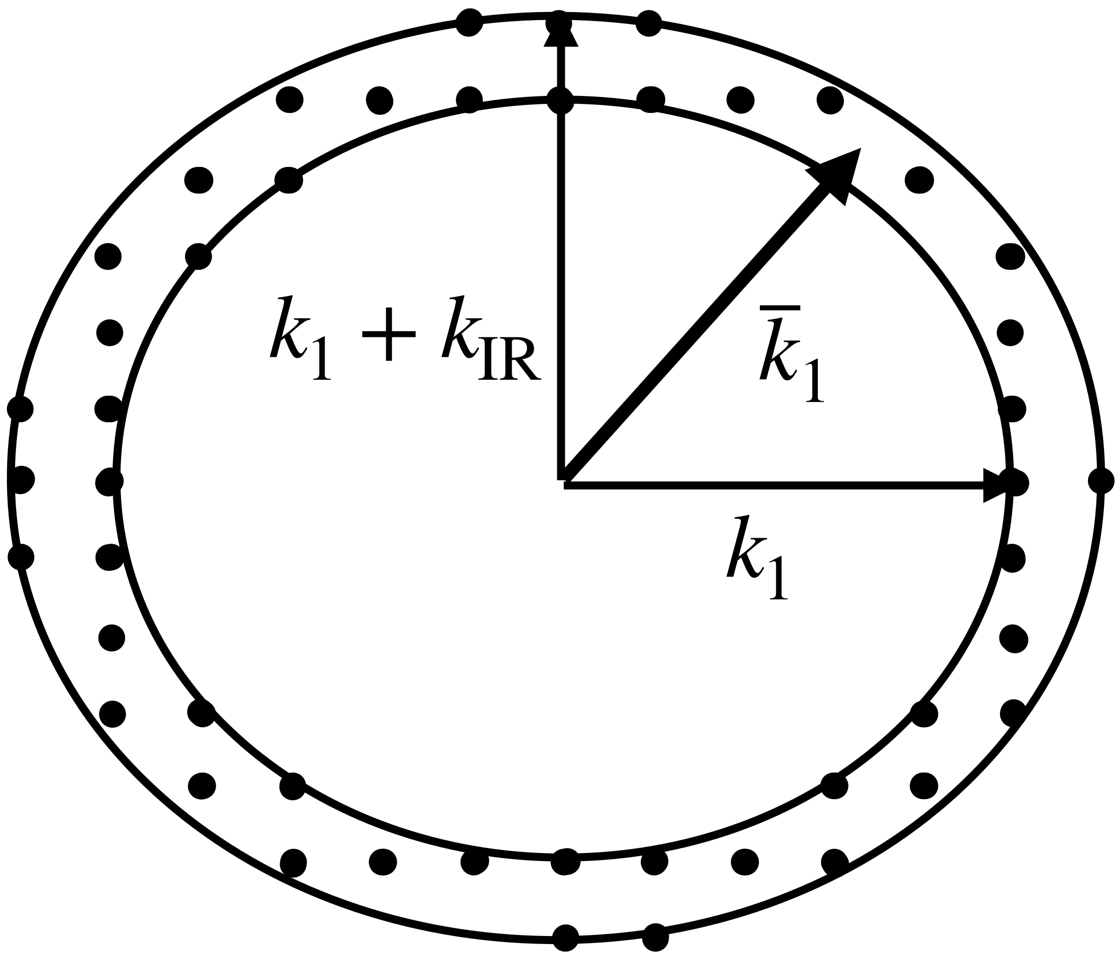
~ works Badly

Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2\right\rangle_{R(\tilde{\mathbf{n}})}$$

$\left(\begin{array}{c}\textbf{Multiplicity} \\ \simeq 4\pi|\tilde{\mathbf{n}}|^2\end{array}\right)$

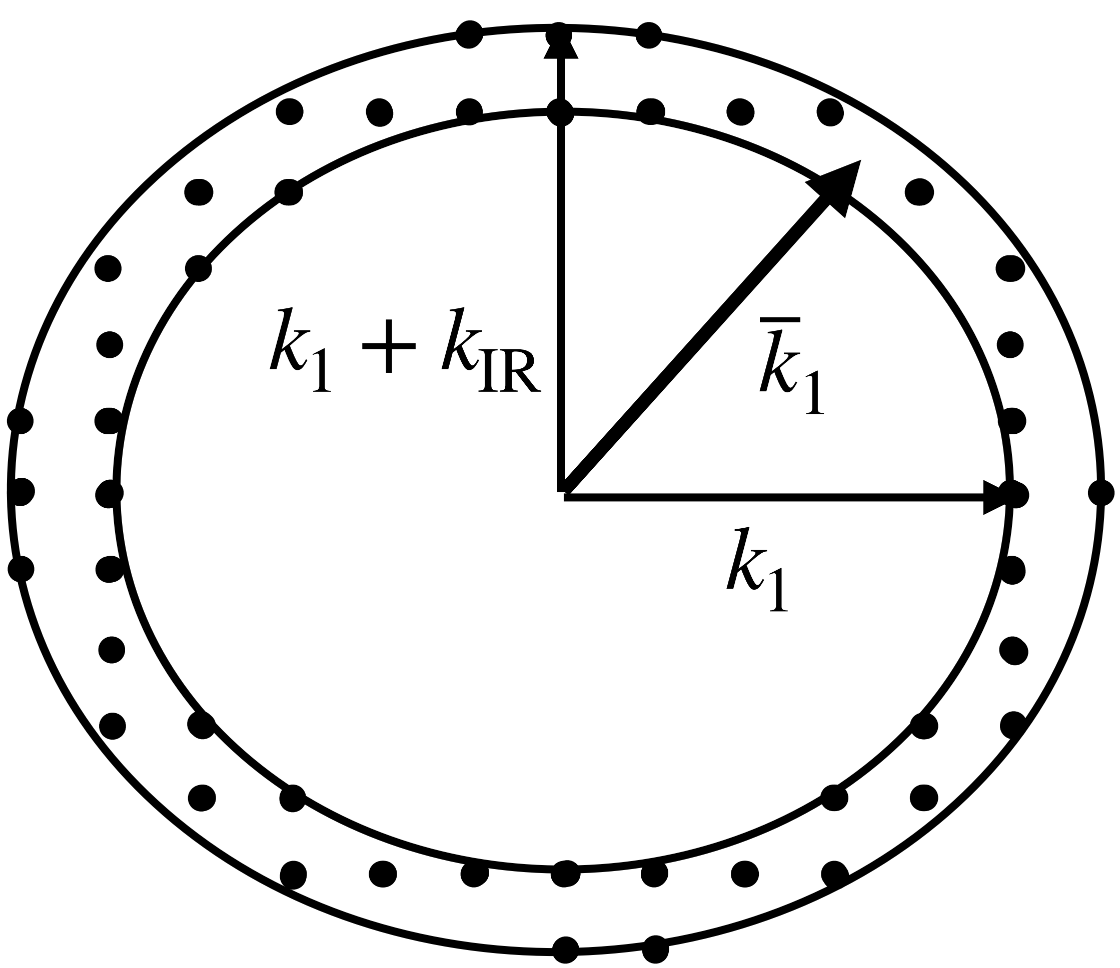


Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \quad \textbf{?}$$

$\left(\begin{array}{c} \#_{|\tilde{\mathbf{n}}|} \textbf{ Real} \\ \textbf{Multiplicity} \end{array} \right)$



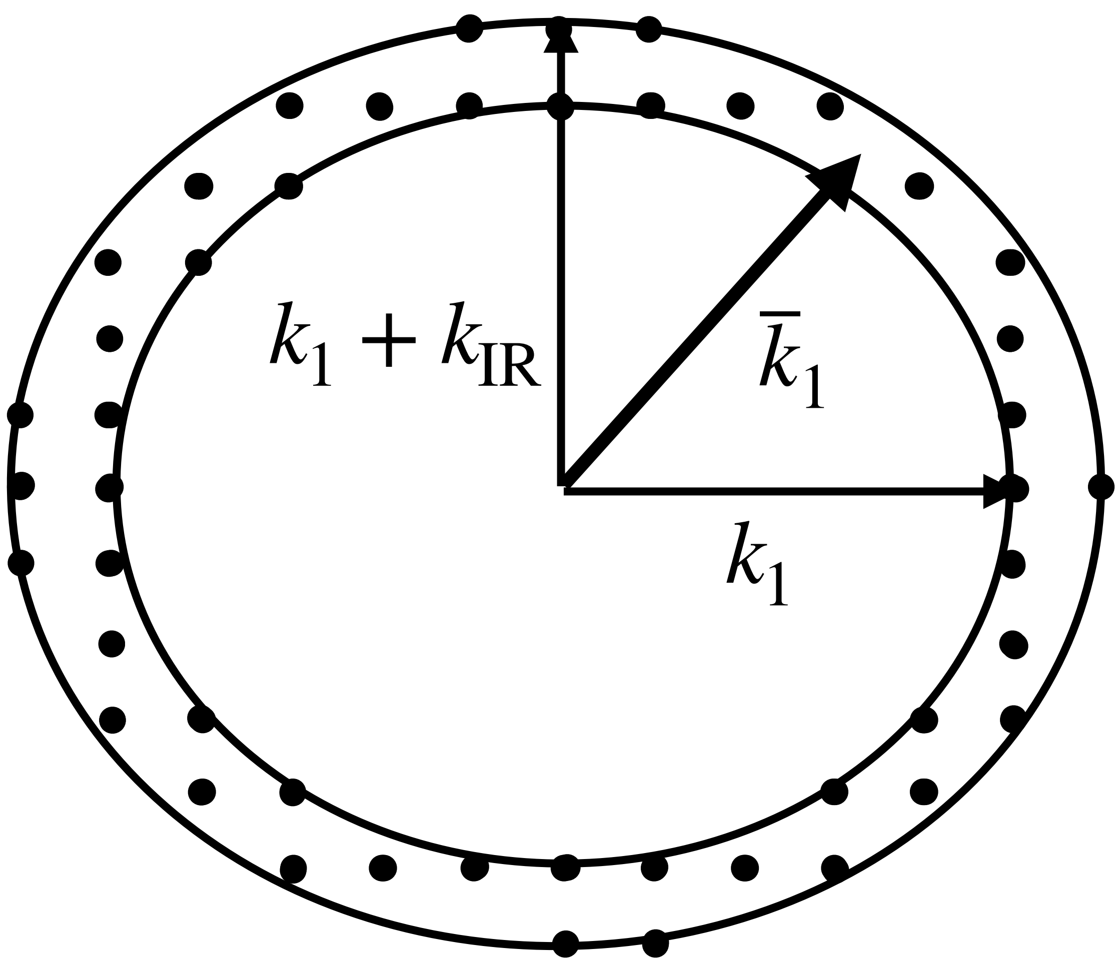
Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

$\left(\#_{|\tilde{\mathbf{n}}|}\right.$ **Real Multiplicity** $\left.\right)$

(Similar calculation as before)



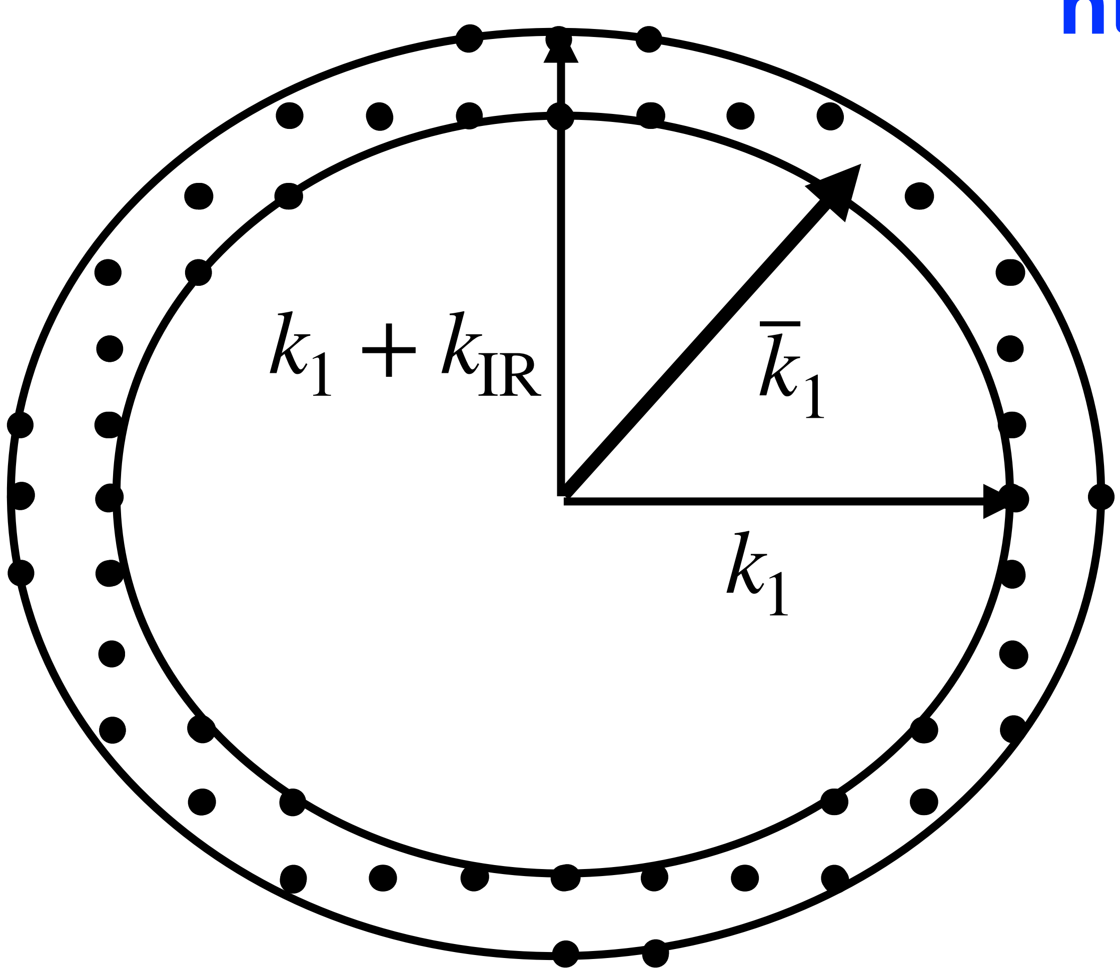
Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\, \#_{|\tilde{\mathbf{n}}|} \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} \qquad \left(\#_{|\tilde{\mathbf{n}}|} \text{ Real Multiplicity} \right)$$

(Similar calculation as before)

See Technical Note I, on the definition of Power Spectrum
https://cosmolattice.net/technicalnotes/

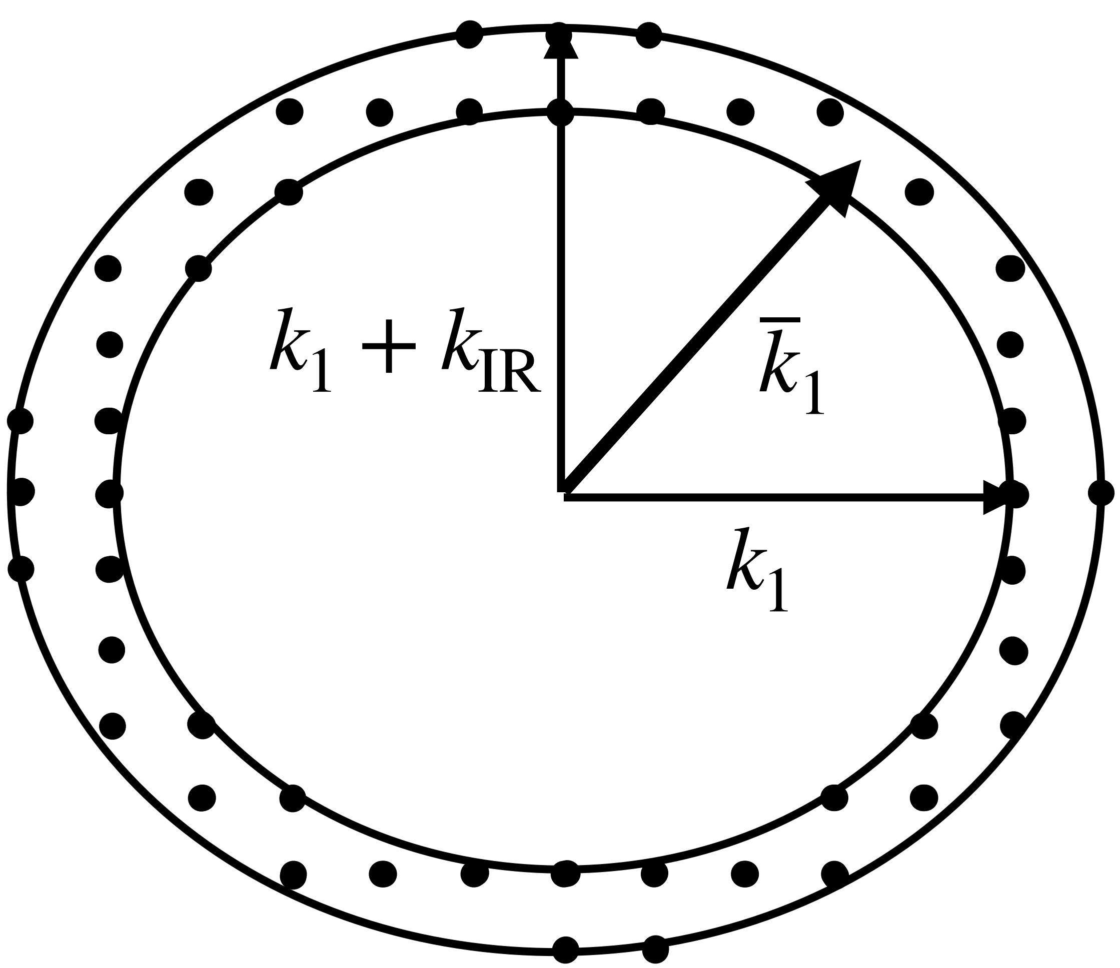


Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} \qquad \left(\#_{|\tilde{\mathbf{n}}|}\ \textbf{Real Multiplicity} \right)$$

(Similar calculation as before)



Primer on Lattice Techniques

Definition of Power Spectrum

$$\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

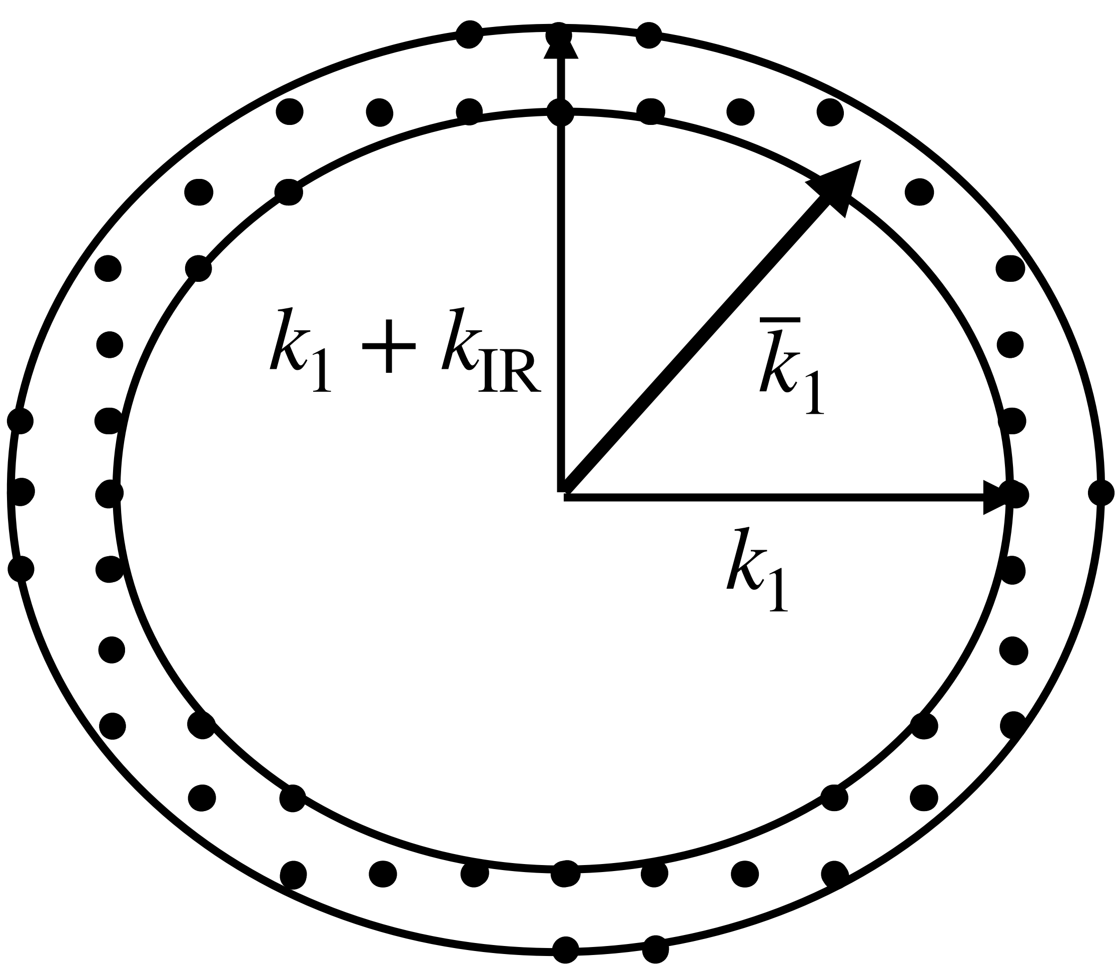
($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

If: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$

(sub-Nyquist freq.'s)

$$\simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)



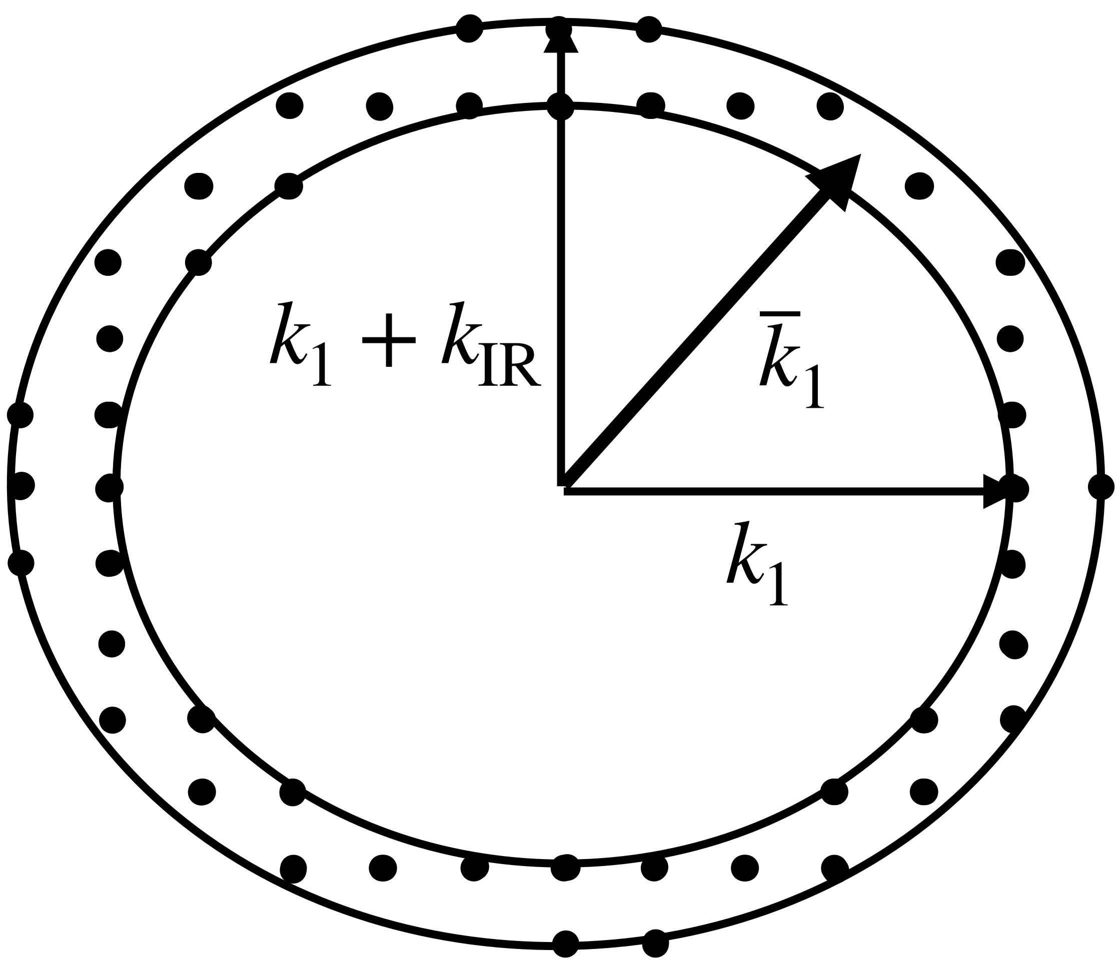
Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS: (Correct for all lattice site)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)



Primer on Lattice Techniques

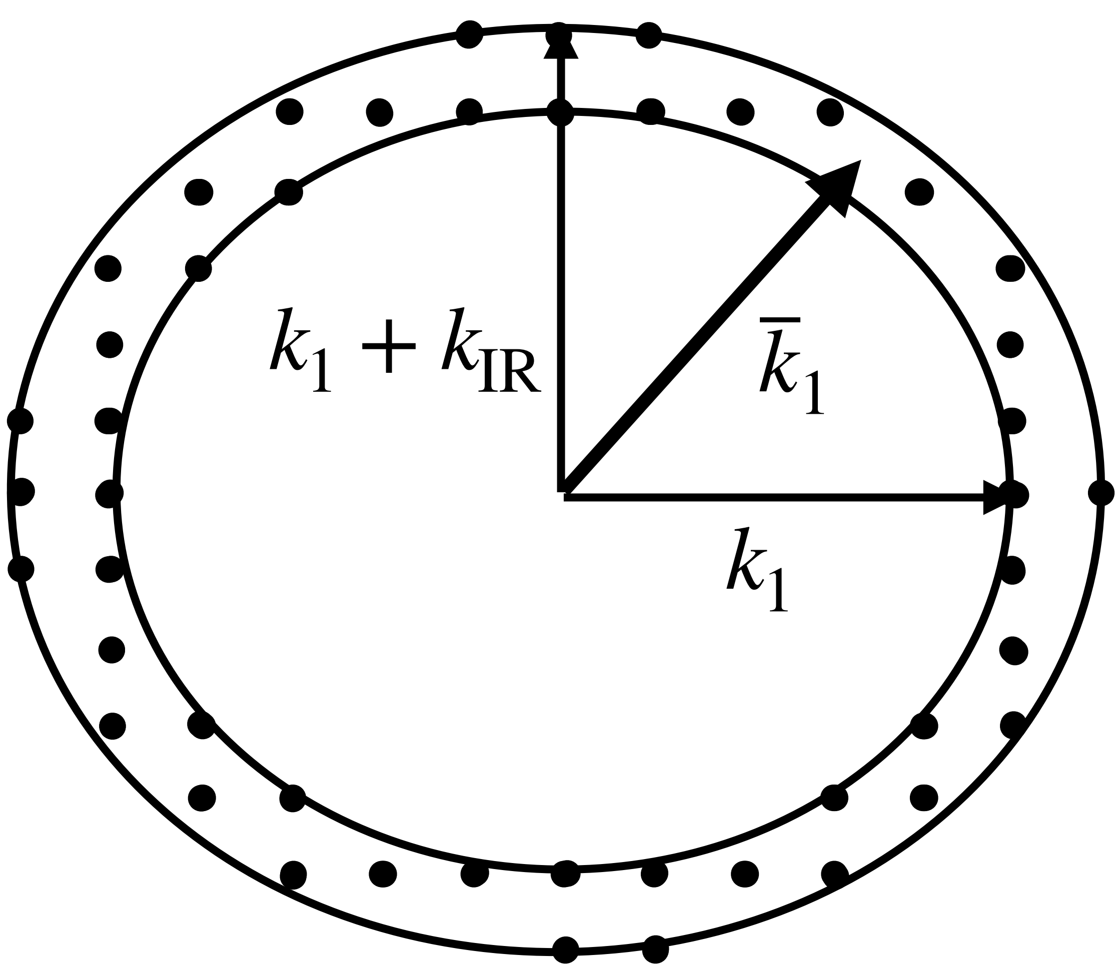
Definition of Power Spectrum

Type-I Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

(**Correct for all lattice site**)

Type-II Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ (**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

(**Correct for Sub-Nyquist modes only:** $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)



Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS: $\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

(**Correct for all lattice site**)

Type-II Lattice PS: $\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ (**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

(**Correct for Sub-Nyquist modes only:** $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS:

(Correct for all lattice site)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

Type-II Lattice PS:

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left| \mathfrak{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

CosmoLattice → **Choose:** ***Type*** **+** ***Version*** **+** ***Verbosity***

Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5} \#_{|\tilde{\mathbf{n}}|} \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

(**Correct for all lattice site**)

Type-II Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2} \frac{L^3}{N^6} \left\langle \left| \mathfrak{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ (**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

(**Correct for Sub-Nyquist modes only:** $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

CosmoLattice → **Choose:** ***Type*** **+** ***Version*** **+** ***Verbosity***

e.g. Type I

Mean bin momentum: $k(l) = \frac{1}{2}(k_{\max}^{(l)} + k_{\min}^{(l)}(l))$

Weighted bin momentum: $\langle k(\tilde{\mathbf{n}}')\rangle_l \equiv \frac{k_{\mathrm{IR}}}{\#_l} \sum_{\tilde{\mathbf{n}}' \in R(l)} |\tilde{\mathbf{n}}'|$

Version

1: $\Delta_f^{(\mathrm{I})}(l) = \frac{k(l)\delta x}{2\pi N^5} \#_l \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(l)}$

2: $\Delta_f^{(\mathrm{I})}(l) = \frac{\langle k(\tilde{\mathbf{n}}')\rangle_l \delta x}{2\pi N^5} \#_l \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(l)}$

3: $\Delta_f^{(\mathrm{I})}(l) = \frac{\delta x \#_l}{2\pi N^5} \left\langle k(\tilde{\mathbf{n}}') \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(l)}$

Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ **Real Multiplicity**)

(**Correct for all lattice site**)

Type-II Lattice PS: $\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathfrak{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ (**Multiplicity** $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

(**Correct for Sub-Nyquist modes only:** $1 \lesssim |\tilde{\mathbf{n}}| \lesssim N/2$)

CosmoLattice → **Choose:** ***Type*** **+** ***Version*** **+** ***Verbosity*** **e.g. Type I**

Verbosity		***Output***
0: $k(l) = \frac{1}{2}(k_{\rm max}^{(l)} + k_{\rm min}^{(l)}(l))$	⇒	$\{\, k(l),\ \Delta_f^{(I)}(l) \,\}$
1: $\langle k(\tilde{\mathbf{n}}')\rangle_l \equiv \frac{k_{\rm IR}}{\#_l}\sum_{\tilde{\mathbf{n}}'\in R(l)} |\tilde{\mathbf{n}}'|$	⇒	$\{\, \langle k(\tilde{\mathbf{n}}')\rangle_l,\ \Delta_f^{(I)}(l) \,\}$
2: ALL	⇒	$\{\, k(l),\ \langle k(\tilde{\mathbf{n}}')\rangle_l,\ rms(k(\tilde{\mathbf{n}}')),\ k_{\rm min}^{(l)},\ k_{\rm max}^{(l)}$ $\Delta_f^{(I)}, \langle\Delta_f^{(I)}\rangle, rms(\Delta_f^{(I)}), \Delta_{\rm min}^{(I)}, \Delta_{\rm max}^{(I)} \,\}$

Version

1: $\Delta_f^{(\mathrm{I})}(l) = \frac{k(l)\delta x}{2\pi N^5}\#_l\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

2: $\Delta_f^{(\mathrm{I})}(l) = \frac{\langle k(\tilde{\mathbf{n}}')\rangle_l\delta x}{2\pi N^5}\#_l\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

3: $\Delta_f^{(\mathrm{I})}(l) = \frac{\delta x\#_l}{2\pi N^5}\left\langle k(\tilde{\mathbf{n}}')\left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

Primer on Lattice Techniques

Definition of Power Spectrum

Type-I Lattice PS: $\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

(Correct for all lattice site)

Type-II Lattice PS: $\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathfrak{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ (Multiplicity $\simeq 4\pi|\tilde{\mathbf{n}}|^2$)

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

CosmoLattice → Choose: *Type* + *Version* + *Verbosity* **e.g. Type II**

Verbosity	*Output*
0: $k(l) = \frac{1}{2}(k_{\max}^{(l)} + k_{\min}^{(l)}(l))$	$\{\, k(l),\ \Delta_f^{(II)}(l) \,\}$
1: $\langle k(\tilde{\mathbf{n}}')\rangle_l \equiv \frac{k_{\rm IR}}{\#_l}\sum_{\tilde{\mathbf{n}}'\in R(l)}|\tilde{\mathbf{n}}'|$	$\{\, \langle k(\tilde{\mathbf{n}}')\rangle_l,\ \Delta_f^{(II)}(l) \,\}$
2: ALL	$\{\, k(l),\ \langle k(\tilde{\mathbf{n}}')\rangle_l,\ rms(k(\tilde{\mathbf{n}}')),\ k_{\min}^{(l)},\ k_{\max}^{(l)},\ \Delta_f^{(II)},\ \langle\Delta_f^{(II)}\rangle,\ rms(\Delta_f^{(II)}),\ \Delta_{\min}^{(II)},\ \Delta_{\max}^{(II)} \,\}$

Version

1: $\Delta_f^{(II)}(l) = \frac{k^3(l)}{2\pi^2}\frac{\delta x^3}{N^3}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

2: $\Delta_f^{(II)}(l) = \frac{\langle k(\tilde{\mathbf{n}}')\rangle_l^3}{2\pi^2}\frac{\delta x^3}{N^3}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

3: $\Delta_f^{(II)}(l) = \frac{1}{2\pi^2}\frac{\delta x^3}{N^3}\left\langle k^3(\tilde{\mathbf{n}}')\left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$

Primer on Lattice Techniques

Summary

* Lattice Definition
* Fourier Tranform
* Fourier Lattice (Reciprocal)
* Lattice Derivatives
* Lattice Momenta
* Power Spectrum

— Lecture 1 —

Welcome to the Lattice

* L1.a: Overview of CosmoLattice (CL) ✓
* L1.b: What is really a Lattice ? ✓

CosmoLattice – School 2022

Day 1
(Monday 5th)

- Lesson 1: Welcome to the Lattice — **Dani** ✓
- Lesson 2: Inflation and post-inflationary dynamics — **Paco** (morning)
- Lesson 2b: Primer on Lattice simulations — **Paco** (afternoon)
- Practice — **All together** (afternoon)

CosmoLattice – School 2022

Day 1
(Monday 5th)

- **Lesson 1: Welcome to the Lattice — Dani** ✓
- **Lesson 2: Inflation and post-inflationary dynamics — Paco** (morning)
- **Lesson 2b: Primer on Lattice simulations — Paco** (afternoon)
- **Practice — All together** (afternoon)

Shall we Coffee Break ?